

# GEOMETRY OF THE CIRCLE

Early geometers in many parts of the world knew that, for all circles, the ratio of the circumference of a circle to its diameter was a constant. Today, we write  $\frac{C}{d} = \pi$ , but early geometers did not use the symbol  $\pi$  to represent this constant. Euclid established that the ratio of the area of a circle to the square of its diameter was also a constant, that is,  $\frac{A}{d^2} = k$ . How do these constants,  $\pi$  and  $k$ , relate to one another?

Archimedes (287–212 B.C.) proposed that the area of a circle was equal to the area of a right triangle whose legs have lengths equal to the radius,  $r$ , and the circumference,  $C$ , of a circle. Thus  $A = \frac{1}{2}rC$ . He used indirect proof and the areas of inscribed and circumscribed polygons to prove his conjecture and to prove that  $3\frac{10}{71} < \pi < 3\frac{1}{7}$ . Since this inequality can be written as  $3.140845 \dots < \pi < 3.142857 \dots$ , Archimedes' approximation was correct to two decimal places.

Use Archimedes' formula for the area of a circle and the facts that  $\frac{C}{d} = \pi$  and  $d = 2r$  to show that  $A = \pi r^2$  and that  $4k = \pi$ .

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## 13-1 ARCS AND ANGLES

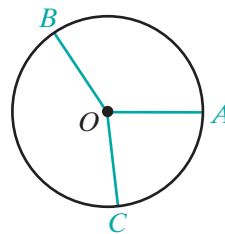
In Chapter 11, we defined a sphere and found that the intersection of a plane and a sphere was a circle. In this chapter, we will prove some important relationships involving the measures of angles and line segments associated with circles. Recall the definition of a circle.

### DEFINITION

A **circle** is the set of all points in a plane that are equidistant from a fixed point of the plane called the **center** of the circle.

If the center of a circle is point  $O$ , the circle is called circle  $O$ , written in symbols as  $\odot O$ .

A **radius** of a circle (plural, *radii*) is a line segment from the center of the circle to any point of the circle. The term *radius* is used to mean both the line segment and the length of the line segment. If  $A$ ,  $B$ , and  $C$  are points of circle  $O$ , then  $\overline{OA}$ ,  $\overline{OB}$ , and  $\overline{OC}$  are radii of the circle. Since the definition of a circle states that all points of the circle are equidistant from its center,  $OA = OB = OC$ . Thus,  $\overline{OA} \cong \overline{OB} \cong \overline{OC}$  because equal line segments are congruent. We can state what we have just proved as a theorem.

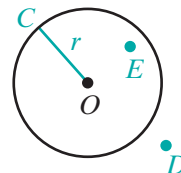


### Theorem 13.1

All radii of the same circle are congruent.

A circle separates a plane into three sets of points. If we let the length of the radius of circle  $O$  be  $r$ , then:

- Point  $C$  is on the circle if  $OC = r$ .
- Point  $D$  is outside the circle if  $OD > r$ .
- Point  $E$  is inside the circle if  $OE < r$ .



The **interior of a circle** is the set of all points whose distance from the center of the circle is less than the length of the radius of the circle.

The **exterior of a circle** is the set of all points whose distance from the center of the circle is greater than the length of the radius of the circle.

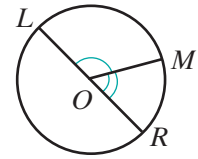
## Central Angles

Recall that an angle is the union of two rays having a common endpoint and that the common endpoint is called the vertex of the angle.

### DEFINITION

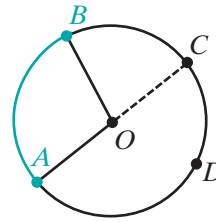
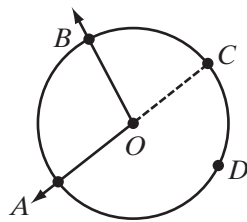
A **central angle of a circle** is an angle whose vertex is the center of the circle.

In the diagram,  $\angle LOM$  and  $\angle MOR$  are central angles because the vertex of each angle is point  $O$ , the center of the circle.

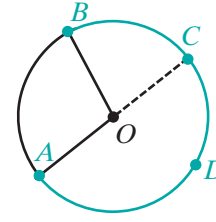


## Types of Arcs

An **arc of a circle** is the part of the circle between two points on the circle. In the diagram,  $A$ ,  $B$ ,  $C$ , and  $D$  are points on circle  $O$  and  $\angle AOB$  intersects the circle at two distinct points,  $A$  and  $B$ , separating the circle into two arcs.



Minor arc  $(\widehat{AB})$



Major arc  $(\widehat{ACB})$

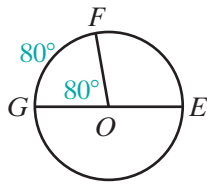
1. If  $m\angle AOB < 180$ , points  $A$  and  $B$  and the points of the circle in the interior of  $\angle AOB$  make up **minor arc  $AB$** , written as  $\widehat{AB}$ .
2. Points  $A$  and  $B$  and the points of the circle not in the interior of  $\angle AOB$  make up **major arc  $AB$** . A major arc is usually named by three points: the two endpoints and any other point on the major arc. Thus, the major arc with endpoints  $A$  and  $B$  is written as  $\widehat{ACB}$  or  $\widehat{ADB}$ .
3. If  $m\angle AOC = 180$ , points  $A$  and  $C$  separate circle  $O$  into two equal parts, each of which is called a **semicircle**. In the diagram above,  $\widehat{ADC}$  and  $\widehat{ABC}$  name two different semicircles.

## Degree Measure of an Arc

An arc of a circle is called an **intercepted arc**, or an arc intercepted by an angle, if each endpoint of the arc is on a different ray of the angle and the other points of the arc are in the interior of the angle.

### DEFINITION

The **degree measure of an arc** is equal to the measure of the central angle that intercepts the arc.



In circle  $O$ ,  $\angle GOE$  is a straight angle,  $m\angle GOE = 180$ , and  $m\angle FOG = 80$ . Therefore, the degree measure of  $\widehat{FG}$  is  $80^\circ$ , written as  $m\widehat{FG} = 80$ . Also, since  $m\widehat{GFE} = 180$ ,

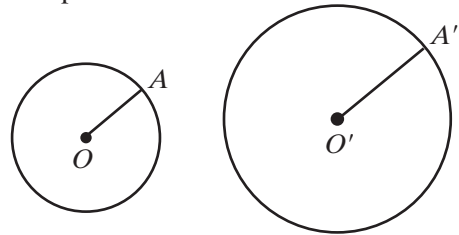
$$\begin{aligned} m\widehat{FE} &= 180 - 80 & \text{and} & & m\widehat{FEG} &= 100 + 180 \\ &= 100 & & & &= 280 \end{aligned}$$

Therefore,

$$\begin{aligned} m\widehat{GF} + m\widehat{FEG} &= 80 + 280 \\ &= 360 \end{aligned}$$

1. The degree measure of a semicircle is 180. Thus:
2. The degree measure of a major arc is equal to 360 minus the degree measure of the minor arc having the same endpoints.

Do not confuse the degree measure of an arc with the length of an arc. The degree measure of every circle is 360 but the circumference of a circle is  $2\pi$  times the radius of the circle. Example: in circle  $O$ ,  $OA = 1$  cm, and in circle  $O'$ ,  $O'A' = 1.5$  cm. In



both circles, the degree measure of the circle is  $360^\circ$  but the circumference of circle  $O$  is  $2\pi$  centimeters, and the circumference of circle  $O'$  is  $3\pi$  centimeters.

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## Congruent Circles, Congruent Arcs, and Arc Addition

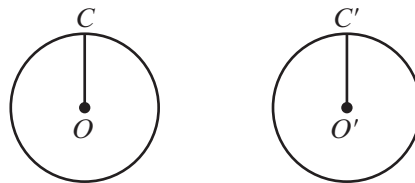
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### DEFINITION

**Congruent circles** are circles with congruent radii.

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If  $\overline{OC} \cong \overline{O'C'}$ , then circles  $O'$  and  $O$  are congruent.

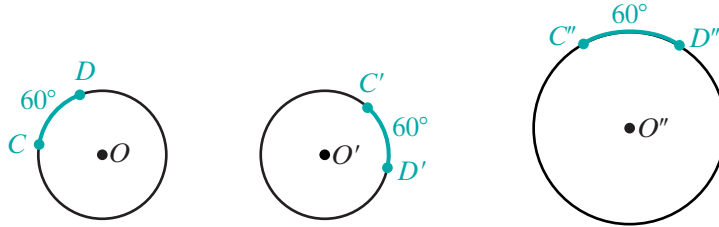


### DEFINITION

**Congruent arcs** are arcs of the same circle or of congruent circles that are equal in measure.

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If  $\odot O \cong \odot O'$  and  $m\widehat{CD} = m\widehat{C'D'} = 60$ , then  $\widehat{CD} \cong \widehat{C'D'}$ . However, if circle  $O$  is *not* congruent to circle  $O''$ , then  $\widehat{CD}$  is *not* congruent to  $\widehat{C''D''}$  even if  $\widehat{CD}$  and  $\widehat{C''D''}$  have the same degree measure.


**Postulate 13.1**
**Arc Addition Postulate**

If  $\widehat{AB}$  and  $\widehat{BC}$  are two arcs of the same circle having a common endpoint and no other points in common, then  $\widehat{AB} + \widehat{BC} = \widehat{ABC}$  and  $m\widehat{AB} + m\widehat{BC} = m\widehat{ABC}$ .

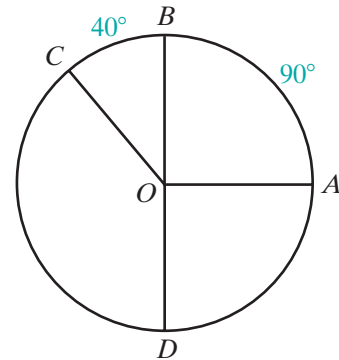
The arc that is the sum of two arcs may be a minor arc, a major arc, or semicircle. For example,  $A$ ,  $B$ ,  $C$ , and  $D$  are points of circle  $O$ ,  $m\widehat{AB} = 90$ ,  $m\widehat{BC} = 40$ , and  $\overrightarrow{OB}$  and  $\overrightarrow{OD}$  are opposite rays.

1. *Minor arc:*  $\widehat{AB} + \widehat{BC} = \widehat{AC}$

$$\begin{aligned} \text{Also, } m\widehat{AC} &= m\widehat{AB} + m\widehat{BC} \\ &= 90 + 40 \\ &= 130 \end{aligned}$$

2. *Major arc:*  $\widehat{AB} + \widehat{BCD} = \widehat{ABD}$

$$\begin{aligned} \text{Also, } m\widehat{ABD} &= m\widehat{AB} + m\widehat{BCD} \\ &= 90 + 180 \\ &= 270 \end{aligned}$$



3. *Semicircle:* Since  $\overrightarrow{OB}$  and  $\overrightarrow{OD}$  are opposite rays,  $\angle BOD$  is a straight angle. Thus,  $\widehat{BC} + \widehat{CD} = \widehat{BCD}$ , a semicircle. Also,  $m\widehat{BC} + m\widehat{CD} = m\widehat{BCD} = 180$ .

**Theorem 13.2a**

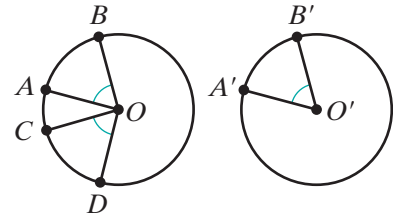
In a circle or in congruent circles, if central angles are congruent, then their intercepted arcs are congruent

**Given** Circle  $O \cong$  circle  $O'$ ,  $\angle AOB \cong \angle COD$ ,  
and  $\angle A'O'B' \cong \angle A'O'B'$ .

**Prove**  $\widehat{AB} \cong \widehat{CD}$  and  $\widehat{AB} \cong \widehat{A'B'}$ .

**Proof** It is given that  $\angle AOB \cong \angle COD$  and  
 $\angle A'O'B' \cong \angle A'O'B'$ . Therefore,

$m\angle AOB = m\angle COD = m\angle A'O'B'$  because congruent angles have equal measures. Then since the degree measure of an arc is equal to the degree measure of the central angle that intercepts that arc,  $m\widehat{AB} = m\widehat{CD} = m\widehat{A'B'}$ . It is also given that circle  $O$  and circle  $O'$  are congruent circles. Congruent arcs are defined as arcs of the same circle or of congruent circles that are equal in measure. Therefore, since their measures are equal,  $\widehat{AB} \cong \widehat{CD}$  and  $\widehat{AB} \cong \widehat{A'B'}$ . ■



The converse of this theorem can be proved by using the same definitions and postulates.

**Theorem 13.2b**

In a circle or in congruent circles, central angles are congruent if their intercepted arcs are congruent.

Theorems 13.2a and 13.2b can be written as a biconditional.

**Theorem 13.2**

In a circle or in congruent circles, central angles are congruent if and only if their intercepted arcs are congruent.

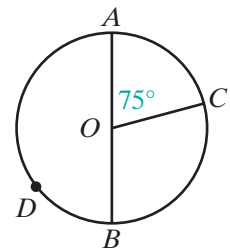
**EXAMPLE 1**

Let  $\overrightarrow{OA}$  and  $\overrightarrow{OB}$  be opposite rays and  $m\angle AOC = 75$ . Find:

- a.  $m\angle BOC$    b.  $m\widehat{AC}$    c.  $m\widehat{BC}$    d.  $m\widehat{AB}$    e.  $m\widehat{BAC}$

**Solution**

$$\begin{aligned} \text{a. } m\angle BOC &= m\angle AOB - m\angle AOC \\ &= 180 - 75 \\ &= 105 \end{aligned}$$



$$\text{b. } m\widehat{AC} = m\angle AOC = 75$$

$$\text{c. } m\widehat{BC} = m\angle BOC = 105$$

$$\text{d. } m\widehat{AB} = m\angle AOB = 180$$

$$\begin{aligned} \text{e. } m\widehat{BAC} &= m\widehat{BDA} + m\widehat{AC} & \text{or} & & m\widehat{BAC} &= 360 - m\widehat{BC} \\ &= 180 + 75 & & & &= 360 - 105 \\ &= 255 & & & &= 255 \end{aligned}$$

**Answers** a. 105 b. 75 c. 105 d. 180 e. 255 ■

## Exercises

### Writing About Mathematics

1. Kay said that if two lines intersect at the center of a circle, then they intercept two pairs of congruent arcs. Do you agree with Kay? Justify your answer.
2. Four points on a circle separate the circle into four congruent arcs:  $\widehat{AB}$ ,  $\widehat{BC}$ ,  $\widehat{CD}$ , and  $\widehat{DA}$ . Is it true that  $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$ ? Justify your answer.

### Developing Skills

In 3–7, find the measure of the central angle that intercepts an arc with the given degree measure.

3. 35

4. 48

5. 90

6. 140

7. 180

In 8–12, find the measure of the arc intercepted by a central angle with the given measure.

8. 60

9. 75

10. 100

11. 120

12. 170

In 13–22, the endpoints of  $\overline{AOC}$  are on circle  $O$ ,  $m\angle AOB = 89$ , and  $m\angle COD = 42$ . Find each measure.

13.  $m\angle BOC$

15.  $m\widehat{BC}$

17.  $m\widehat{DA}$

19.  $m\widehat{BCD}$

21.  $m\angle AOC$

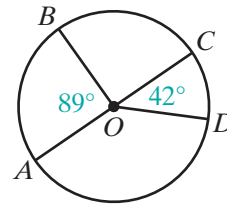
14.  $m\widehat{AB}$

16.  $m\angle DOA$

18.  $m\angle BOD$

20.  $m\widehat{DAB}$

22.  $m\widehat{ADC}$



In 23–32,  $P$ ,  $Q$ ,  $S$ , and  $R$  are points on circle  $O$ ,  $m\angle POQ = 100$ ,  $m\angle QOS = 110$ , and  $m\angle SOR = 35$ . Find each measure.

23.  $m\widehat{PQ}$

25.  $m\widehat{SR}$

27.  $m\widehat{RP}$

29.  $m\angle QOR$

31.  $m\widehat{SRP}$

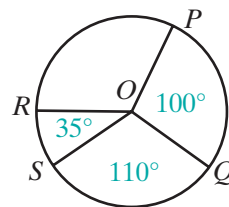
24.  $m\widehat{QS}$

26.  $m\angle ROP$

28.  $m\widehat{PQS}$

30.  $m\widehat{QSR}$

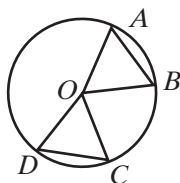
32.  $m\widehat{RPQ}$



### Applying Skills

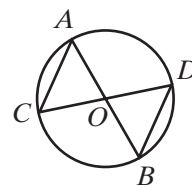
33. *Given:* Circle  $O$  with  $\widehat{AB} \cong \widehat{CD}$ .

*Prove:*  $\triangle AOB \cong \triangle COD$



34. *Given:*  $\overline{AB}$  and  $\overline{CD}$  intersect at  $O$ , and the endpoints of  $\overline{AB}$  and  $\overline{CD}$  are on circle  $O$ .

*Prove:*  $\overline{AC} \cong \overline{BD}$



35. In circle  $O$ ,  $\overline{AOB} \perp \overline{COD}$ . Find  $m\widehat{AC}$  and  $m\widehat{ADC}$ .

36. Points  $A$ ,  $B$ ,  $C$ , and  $D$  lie on circle  $O$ , and  $\overline{AC} \perp \overline{BD}$  at  $O$ . Prove that quadrilateral  $ABCD$  is a square.

### Hands-On Activity



For this activity, you may use compass, protractor, and straightedge, or geometry software.

1. Draw circle  $O$  with a radius of 2 inches and circle  $O'$  with a radius of 3 inches.
2. Draw points  $A$  and  $B$  on the circle  $O$  so that  $m\widehat{AB} = 60$  and points  $A'$  and  $B'$  on circle  $O'$  so that  $m\widehat{A'B'} = 60$ .
3. Show that  $\triangle AOB \sim \triangle A'O'B'$ .



**13-2 ARCS AND CHORDS**
**DEFINITION**

A **chord** of a circle is a line segment whose endpoints are points of the circle.

A **diameter** of a circle is a chord that has the center of the circle as one of its points.

In the diagram,  $\overline{AB}$  and  $\overline{AOC}$  are chords of circle  $O$ . Since  $O$  is a point of  $\overline{AOC}$ ,  $\overline{AOC}$  is a diameter. Since  $OA$  and  $OC$  are the lengths of the radius of circle  $O$ ,  $OA = OC$  and  $O$  is the midpoint of  $\overline{AOC}$ .

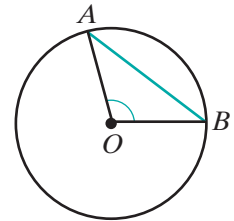
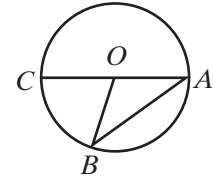
If the length of the radius of circle  $O$  is  $r$ , and the length of the diameter is  $d$ , then

$$\begin{aligned} d &= \overline{AOC} \\ &= \overline{OA} + \overline{OC} \\ &= r + r \\ &= 2r \end{aligned}$$

That is:

$$d = 2r$$

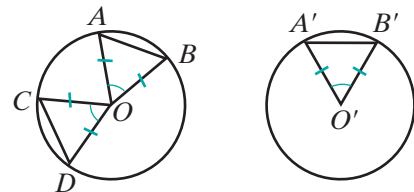
The endpoints of a chord are points on a circle and, therefore, determine two arcs of a circle, a minor arc and a major arc. In the diagram, chord  $\overline{AB}$ , central  $\angle AOB$ , minor  $\widehat{AB}$ , and major  $\widehat{AB}$  are all determined by points  $A$  and  $B$ . We proved in the previous section that in a circle, congruent central angles intercept congruent arcs. Now we can prove that in a circle, congruent central angles have congruent chords and that congruent arcs have congruent chords.


**Theorem 13.3a**

In a circle or in congruent circles, congruent central angles have congruent chords.

**Given**  $\odot O \cong \odot O'$  and  
 $\angle COD \cong \angle AOB \cong \angle A'O'B'$

**Prove**  $\overline{CD} \cong \overline{AB} \cong \overline{A'B'}$



**Proof** We will show that  $\triangle COD \cong \triangle AOB \cong \triangle A'O'B'$  by SAS.

It is given that  $\angle AOB \cong \angle COD$  and  $\angle AOB \cong \angle A'O'B'$ . Therefore,  $\angle AOB \cong \angle COD \cong \angle A'O'B'$  by the transitive property of congruence. Since  $\overline{DO}$ ,  $\overline{CO}$ ,  $\overline{AO}$ ,  $\overline{BO}$ ,  $\overline{A'O'}$ , and  $\overline{B'O'}$  are the radii of congruent circles, these segments are all congruent:

$$\overline{DO} \cong \overline{CO} \cong \overline{AO} \cong \overline{BO} \cong \overline{A'O'} \cong \overline{B'O'}$$

Therefore, by SAS,  $\triangle COD \cong \triangle AOB \cong \triangle A'O'B'$ . Since corresponding parts of congruent triangles are congruent,  $\overline{CD} \cong \overline{AB} \cong \overline{A'B'}$ . ▀

The converse of this theorem is also true.

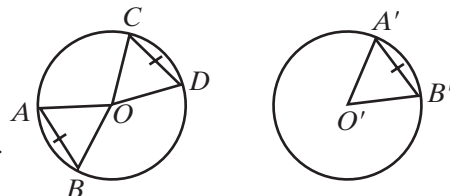
### Theorem 13.3b

In a circle or in congruent circles, congruent chords have congruent central angles.

**Given**  $\odot O \cong \odot O'$  and  $\overline{CD} \cong \overline{AB} \cong \overline{A'B'}$

**Prove**  $\angle COD \cong \angle AOB \cong \angle A'O'B'$

**Strategy** This theorem can be proved in a manner similar to Theorem 13.3a: prove that  $\triangle COD \cong \triangle AOB \cong \triangle A'O'B'$  by SSS.



The proof of Theorem 13.3b is left to the student. (See exercise 23.) Theorems 13.3a and 13.3b can be stated as a biconditional.

### Theorem 13.3

In a circle or in congruent circles, two chords are congruent if and only if their central angles are congruent.

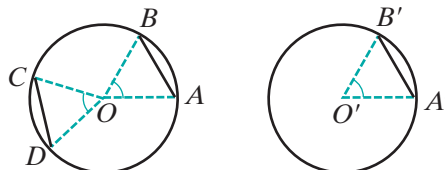
Since central angles and their intercepted arcs have equal degree measures, we can also prove the following theorems.

### Theorem 13.4a

In a circle or in congruent circles, congruent arcs have congruent chords.

**Given**  $\odot O \cong \odot O'$  and  $\widehat{CD} \cong \widehat{AB} \cong \widehat{A'B'}$

**Prove**  $\overline{CD} \cong \overline{AB} \cong \overline{A'B'}$



**Proof** First draw line segments from  $O$  to  $A, B, C, D, A',$  and  $B'$ . Congruent arcs have congruent central angles. Therefore,  $\angle COD \cong \angle AOB \cong \angle A'O'B'$ . In a circle or in congruent circles, two chords are congruent if and only if their central angles are congruent. Therefore,  $\overline{CD} \cong \overline{AB} \cong \overline{A'B'}$ . ■

The converse of this theorem is also true.

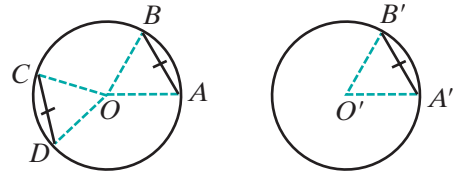
**Theorem 13.4b**

In a circle or in congruent circles, congruent chords have congruent arcs.

**Given**  $\odot O \cong \odot O'$  and  $\overline{CD} \cong \overline{AB} \cong \overline{A'B'}$

**Prove**  $\widehat{CD} \cong \widehat{AB} \cong \widehat{A'B'}$

**Strategy** First draw line segments from  $O$  to  $A, B, C, D, A'$  and  $B'$ .



Prove  $\triangle COD \cong \triangle AOB \cong \triangle A'O'B'$  by SSS. Then use congruent central angles to prove congruent arcs.

The proof of Theorem 13.4b is left to the student. (See exercise 24.) Theorems 13.4a and 13.4b can be stated as a biconditional.

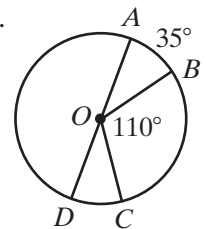
**Theorem 13.4**

In a circle or in congruent circles, two chords are congruent if and only if their arcs are congruent.

**EXAMPLE 1**

In circle  $O$ ,  $m\widehat{AB} = 35$ ,  $m\angle BOC = 110$ , and  $\overline{AOD}$  is a diameter.

- Find  $m\widehat{BC}$  and  $m\widehat{CD}$ .
- Explain why  $AB = CD$ .



**Solution**

$\begin{aligned} \text{a. } m\widehat{BC} &= m\angle BOC \\ &= 110 \end{aligned}$	$\begin{aligned} m\widehat{CD} &= 180 - m\widehat{AB} - m\widehat{BC} \\ &= 180 - 35 - 110 \\ &= 35 \end{aligned}$
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- In a circle, arcs with equal measure are congruent. Therefore, since  $m\widehat{AB} = 35$  and  $m\widehat{CD} = 35$ ,  $\widehat{AB} \cong \widehat{CD}$ . In a circle, chords are congruent if their arcs are congruent. Therefore,  $\overline{AB} \cong \overline{CD}$  and  $AB = CD$ . ■

## Chords Equidistant from the Center of a Circle

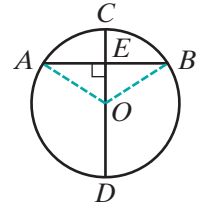
We defined the distance from a point to a line as the length of the perpendicular from the point to the line. The perpendicular is the shortest line segment that can be drawn from a point to a line. These facts can be used to prove the following theorem.

### Theorem 13.5

A diameter perpendicular to a chord bisects the chord and its arcs.

*Given* Diameter  $\overline{COD}$  of circle  $O$ , chord  $\overline{AB}$ , and  $\overline{AB} \perp \overline{CD}$  at  $E$ .

*Prove*  $\overline{AE} \cong \overline{BE}$ ,  $\widehat{AC} \cong \widehat{BC}$ , and  $\widehat{AD} \cong \widehat{BD}$ .



*Proof*

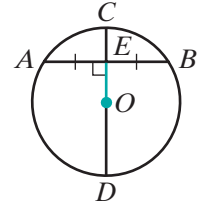
	Statements	Reasons
	1. Draw $\overline{OA}$ and $\overline{OB}$ .	1. Two points determine a line.
	2. $\overline{AB} \perp \overline{CD}$	2. Given.
	3. $\angle AEO$ and $\angle BEO$ are right angles.	3. Perpendicular lines intersect to form right angles.
	4. $\overline{OA} \cong \overline{OB}$	4. Radii of a circle are congruent.
	5. $\overline{OE} \cong \overline{OE}$	5. Reflexive property of congruence.
	6. $\triangle AOE \cong \triangle BOE$	6. HL.
	7. $\overline{AE} \cong \overline{BE}$	7. Corresponding parts of congruent triangles are congruent.
	8. $\angle AOE \cong \angle BOE$	8. Corresponding parts of congruent triangles are congruent.
	9. $\widehat{AC} \cong \widehat{BC}$	9. In a circle, congruent central angles have congruent arcs.
	10. $\angle AOD$ is the supplement of $\angle AOE$ . $\angle BOD$ is the supplement of $\angle BOE$ .	10. If two angles form a linear pair, then they are supplementary.
	11. $\angle AOD \cong \angle BOD$	11. Supplements of congruent angles are congruent.
	12. $\widehat{AD} \cong \widehat{BD}$	12. In a circle, congruent central angles have congruent arcs. <span style="float: right;">■</span>

Since a diameter is a segment of a line, the following corollary is also true:

**Corollary 13.5a**

A line through the center of a circle that is perpendicular to a chord bisects the chord and its arcs.

An **apothem** of a circle is a perpendicular line segment from the center of a circle to the midpoint of a chord. The term *apothem* also refers to the length of the segment. In the diagram,  $E$  is the midpoint of chord  $\overline{AB}$  in circle  $O$ ,  $\overline{AB} \perp \overline{CD}$ , and  $\overline{OE}$ , or  $OE$ , is the apothem.

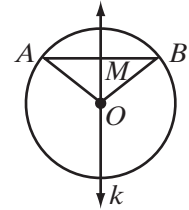

**Theorem 13.6**

The perpendicular bisector of the chord of a circle contains the center of the circle.

**Given** Circle  $O$ , and chord  $\overline{AB}$  with midpoint  $M$  and perpendicular bisector  $k$ .

**Prove** Point  $O$  is a point on  $k$ .

**Proof** In the diagram,  $M$  is the midpoint of chord  $\overline{AB}$  in circle  $O$ . Then,  $AM = MB$  and  $AO = OB$  (since these are radii). Points  $O$  and  $M$  are each equidistant from the endpoints of  $\overline{AB}$ . Two points that are each equidistant from the endpoints of a line segment determine the perpendicular bisector of the line segment. Therefore,  $\overleftrightarrow{OM}$  is the perpendicular bisector of  $\overline{AB}$ . Through a point on a line there is only one perpendicular line. Thus,  $\overleftrightarrow{OM}$  and  $k$  are the same line, and  $O$  is on  $k$ , the perpendicular bisector of  $\overline{AB}$ . ■

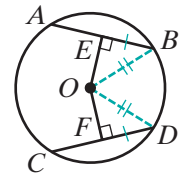

**Theorem 13.7a**

If two chords of a circle are congruent, then they are equidistant from the center of the circle.

**Given** Circle  $O$  with  $\overline{AB} \cong \overline{CD}$ ,  $\overline{OE} \perp \overline{AB}$ , and  $\overline{OF} \perp \overline{CD}$ .

**Prove**  $\overline{OE} \cong \overline{OF}$

**Proof** A line through the center of a circle that is perpendicular to a chord bisects the chord and its arcs. Therefore,  $\overleftrightarrow{OE}$  and  $\overleftrightarrow{OF}$  bisect the congruent chords  $\overline{AB}$  and  $\overline{CD}$ . Since halves of congruent segments are congruent,  $\overline{EB} \cong \overline{FD}$ . Draw  $\overline{OB}$  and  $\overline{OD}$ . Since  $\overline{OB}$  and  $\overline{OD}$  are radii of the same circle,  $\overline{OB} \cong \overline{OD}$ . Therefore,  $\triangle OBE \cong \triangle ODF$  by HL, and  $\overline{OE} \cong \overline{OF}$ . ■



The converse of this theorem is also true.

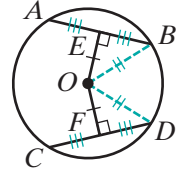
**Theorem 13.7b**

If two chords of a circle are equidistant from the center of the circle, then the chords are congruent.

**Given** Circle  $O$  with  $\overline{OE} \perp \overline{AB}$ ,  $\overline{OF} \perp \overline{CD}$ , and  $\overline{OE} \cong \overline{OF}$ .

**Prove**  $\overline{AB} \cong \overline{CD}$

**Proof** Draw  $\overline{OB}$  and  $\overline{OD}$ . Since  $\overline{OB}$  and  $\overline{OD}$  are radii of the same circle,  $\overline{OB} \cong \overline{OD}$ . Therefore,  $\triangle OBE \cong \triangle ODF$  by HL, and  $\overline{EB} \cong \overline{FD}$ . A line through the center of a circle that is perpendicular to a chord bisects the chord. Thus,  $\overline{AE} \cong \overline{EB}$  and  $\overline{CF} \cong \overline{FD}$ . Since doubles of congruent segments are congruent,  $\overline{AB} \cong \overline{CD}$ . ■



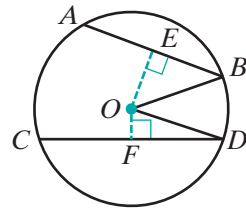
We can state theorems 13.7a and 13.7b as a biconditional.

**Theorem 13.7**

Two chords are equidistant from the center of a circle if and only if the chords are congruent.

What if two chords are not equidistant from the center of a circle? Which is the longer chord? We know that a diameter contains the center of the circle and is the longest chord of the circle. This suggests that the shorter chord is farther from the center of the circle.

- (1) Let  $\overline{AB}$  and  $\overline{CD}$  be two chords of circle  $O$  and  $AB < CD$ .
- (2) Draw  $\overline{OE} \perp \overline{AB}$  and  $\overline{OF} \perp \overline{CD}$ .
- (3) A line through the center of a circle that is perpendicular to the chord bisects the chord.  
Therefore,  $\frac{1}{2}AB = EB$  and  $\frac{1}{2}CD = FD$ .
- (4) The distance from a point to a line is the length of the perpendicular from the point to the line. Therefore,  $OE$  is the distance from the center of the circle to  $\overline{AB}$ , and  $OF$  is the distance from the center of the circle to  $\overline{CD}$ .



*Recall that the squares of equal quantities are equal, that the positive square roots of equal quantities are equal, and that when an inequality is multiplied by a negative number, the inequality is reversed.*

(5) Since  $AB < CD$ :

$$\begin{aligned} AB &< CD \\ \frac{1}{2}AB &< \frac{1}{2}CD \\ EB &< FD \\ EB^2 &< FD^2 \\ -EB^2 &> -FD^2 \end{aligned}$$

(6) Since  $\triangle OBE$  and  $\triangle ODF$  are right triangles and  $OB = OD$ :

$$\begin{aligned} OB^2 &= OD^2 \\ OE^2 + EB^2 &= OF^2 + FD^2 \end{aligned}$$

(7) When equal quantities are added to both sides of an inequality, the order of the inequality remains the same. Therefore, adding the equal quantities from step 6 to the inequality in step 5 gives:

$$\begin{aligned} OE^2 + EB^2 - EB^2 &> OF^2 + FD^2 - FD^2 \\ OE^2 &> OF^2 \\ OE &> OF \end{aligned}$$

Therefore, the shorter chord is farther from the center of the circle. We have just proved the following theorem:

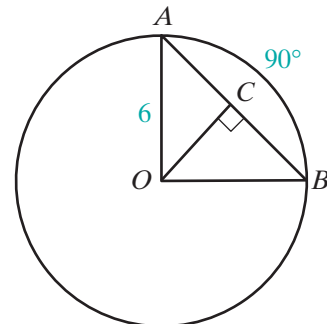
**Theorem 13.8**

In a circle, if the lengths of two chords are unequal, then the shorter chord is farther from the center.

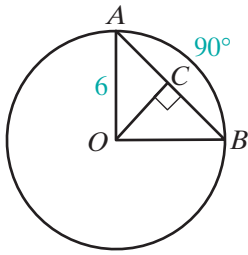
**EXAMPLE 2**

In circle  $O$ ,  $m\widehat{AB} = 90$  and  $OA = 6$ .

- a. Prove that  $\triangle AOB$  is a right triangle.
- b. Find  $AB$ .
- c. Find  $OC$ , the apothem to  $\overline{AB}$ .



- Solution**
- a. If  $m\widehat{AB} = 90$ , then  $m\angle AOB = 90$  because the measure of an arc is equal to the measure of the central angle that intercepts the arc. Since  $\angle AOB$  is a right angle,  $\triangle AOB$  is a right triangle.



- b. Use the Pythagorean Theorem for right  $\triangle AOB$ . Since  $\overline{OA}$  and  $\overline{OB}$  are radii,  $OB = OA = 6$ .

$$AB^2 = OA^2 + OB^2$$

$$AB^2 = 6^2 + 6^2$$

$$AB^2 = 36 + 36$$

$$AB^2 = 72$$

$$AB = \sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$$

- c. Since  $OC$  is the apothem to  $\overline{AB}$ ,  $\overline{OC} \perp \overline{AB}$  and bisects  $\overline{AB}$ . Therefore,  $AC = 3\sqrt{2}$ . In right  $\triangle OCA$ ,

$$OC^2 + AC^2 = OA^2$$

$$OC^2 + (3\sqrt{2})^2 = 6^2$$

$$OC^2 + 18 = 36$$

$$OC^2 = 18$$

$$OC = \sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$$

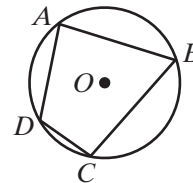
**Note:** In the example, since  $\triangle AOB$  is an isosceles right triangle,  $m\angle AOB$  is 45. Therefore  $\triangle AOC$  is also an isosceles right triangle and  $OC = AC$ .

## Polygons Inscribed in a Circle

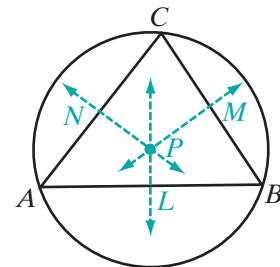
If all of the vertices of a polygon are points of a circle, then the polygon is said to be **inscribed** in the circle. We can also say that the circle is **circumscribed** about the polygon.

In the diagram:

1. Polygon  $ABCD$  is inscribed in circle  $O$ .
2. Circle  $O$  is circumscribed about polygon  $ABCD$ .



In an earlier chapter we proved that the perpendicular bisectors of the sides of a triangle meet at a point and that that point is equidistant from the vertices of the triangle. In the diagram,  $\overleftrightarrow{PL}$ ,  $\overleftrightarrow{PM}$ , and  $\overleftrightarrow{PN}$  are the perpendicular bisectors of the sides of  $\triangle ABC$ . Every point on the perpendicular bisectors of a line segment is equidistant from the endpoints of the line segment. Therefore,  $PA = PB = PC$  and  $A$ ,  $B$ , and  $C$  are points on a circle with center at  $P$ , that is, any triangle can be inscribed in a circle.

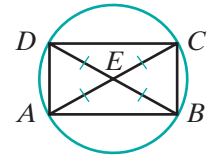




## EXAMPLE 3

Prove that any rectangle can be inscribed in a circle.

**Proof** Let  $ABCD$  be any rectangle. The diagonals of a rectangle are congruent, so  $\overline{AC} \cong \overline{BD}$  and  $AC = BD$ . Since a rectangle is a parallelogram, the diagonals of a rectangle bisect each other. If  $\overline{AC}$  and  $\overline{BD}$  intersect at  $E$ , then  $\frac{1}{2}AC = AE = EC$  and  $\frac{1}{2}BD = BE = ED$ . Halves of equal quantities are equal. Therefore,  $AE = EC = BE = ED$  and the vertices of the rectangle are equidistant from  $E$ . Let  $E$  be the center of a circle with radius  $AE$ . The vertices of  $ABCD$  are on the circle and  $ABCD$  is inscribed in the circle. ■



## Exercises

## Writing About Mathematics

- Daniela said that if a chord is 3 inches from the center of a circle that has a radius of 5 inches, then a 3-4-5 right triangle is formed by the chord, its apothem, and a radius. Additionally, the length of the chord is 4 inches. Do you agree with Daniela? Explain why or why not.
- Two angles that have the same measure are always congruent. Are two arcs that have the same measure always congruent? Explain why or why not.

## Developing Skills

In 3–7, find the length of the radius of a circle whose diameter has the given measure.

3. 6 in.      4. 9 cm      5. 3 ft      6. 24 mm      7.  $\sqrt{24}$  cm

In 8–12, find the length of the diameter of a circle whose radius has the given measure.

8. 5 in.      9. 12 ft      10. 7 cm      11. 6.2 mm      12.  $\sqrt{5}$  yd

13. In circle  $O$ ,  $\overline{AOB}$  is a diameter,  $AB = 3x + 13$ , and  $AO = 2x + 5$ . Find the length of the radius and of the diameter of the circle.

In 14–21,  $\overline{DCOE}$  is a diameter of circle  $O$ ,  $\overline{AB}$  is a chord of the circle, and  $\overline{OD} \perp \overline{AB}$  at  $C$ .

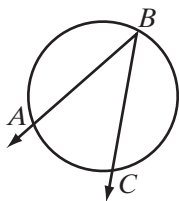
14. If  $AB = 8$  and  $OC = 3$ , find  $OB$ .      15. If  $AB = 48$  and  $OC = 7$ , find  $OB$ .  
 16. If  $OC = 20$  and  $OB = 25$ , find  $AB$ .      17. If  $OC = 12$  and  $OB = 18$ , find  $AB$ .

18. If  $AB = 18$  and  $OB = 15$ , find  $OC$ .
19. If  $AB = 20$  and  $OB = 15$ , find  $OC$ .
20. If  $m\angle AOB = 90$ , and  $AB = 30\sqrt{2}$ , find  $OB$  and  $DE$ .
21. If  $m\angle AOB = 60$ , and  $AB = 30$ , find  $OB$  and  $OC$ .
22. In circle  $O$ , chord  $\overline{LM}$  is 3 centimeters from the center and chord  $\overline{RS}$  is 5 centimeters from the center. Which is the longer chord?

### Applying Skills

23. Prove Theorem 13.3b, “In a circle or in congruent circles, congruent chords have congruent central angles.”
24. Prove Theorem 13.4b, “In a circle or in congruent circles, congruent chords have congruent arcs.”
25. Diameter  $\overline{AOB}$  of circle  $O$  intersects chord  $\overline{CD}$  at  $E$  and bisects  $\widehat{CD}$  at  $B$ . Prove that  $\overline{AOB}$  bisects chord  $\overline{CD}$  and is perpendicular to chord  $\overline{CD}$ .
26. The radius of a spherical ball is 13 centimeters. A piece that has a plane surface is cut off of the ball at a distance of 12 centimeters from the center of the ball. What is the radius of the circular faces of the cut pieces?
27. Triangle  $ABC$  is inscribed in circle  $O$ . The distance from the center of the circle to  $\overline{AB}$  is greater than the distance from the center of the circle to  $\overline{BC}$ , and the distance from the center of the circle to  $\overline{BC}$  is greater than the distance from the center of the circle to  $\overline{AC}$ . Which is the largest angle of  $\triangle ABC$ ? Justify your answer.

## 13-3 INSCRIBED ANGLES AND THEIR MEASURES



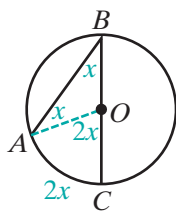
In the diagram,  $\angle ABC$  is an angle formed by two chords that have a common endpoint on the circle.

### DEFINITION

An **inscribed angle** of a circle is an angle whose vertex is on the circle and whose sides contain chords of the circle.

We can use the fact that the measure of a central angle is equal to the measure of its arc to find the relationship between  $\angle ABC$  and the measure of its arc,  $\widehat{AC}$ .

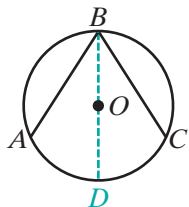
**CASE 1** One of the sides of the inscribed angle contains a diameter of the circle.



Consider first an inscribed angle,  $\angle ABC$ , with  $\overline{BC}$  a diameter of circle  $O$ . Draw  $\overline{OA}$ . Then  $\triangle AOB$  is an isosceles triangle and  $m\angle OAB = m\angle OBA = x$ . Angle  $AOC$  is an exterior angle and  $m\angle AOC = x + x = 2x$ . Since  $\angle AOC$  is a central angle,  $m\angle AOC = m\widehat{AC} = 2x$ . Therefore,  $m\angle ABC = x = \frac{1}{2}m\widehat{AC}$ .

We have shown that when one of the sides of an inscribed angle contains a diameter of the circle, the measure of the inscribed angle is equal to one-half the measure of its intercepted arc. Is this true for angles whose sides do not contain the center of the circle?

**CASE 2** The center of the circle is in the interior of the angle.



Let  $\angle ABC$  be an inscribed angle in which the center of the circle is in the interior of the angle. Draw  $\overline{BOD}$ , a diameter of the circle. Then:

$$m\angle ABD = \frac{1}{2}m\widehat{AD} \quad \text{and} \quad m\angle DBC = \frac{1}{2}m\widehat{DC}$$

Therefore:

$$\begin{aligned} m\angle ABC &= m\angle ABD + m\angle DBC \\ &= \frac{1}{2}m\widehat{AD} + \frac{1}{2}m\widehat{DC} \\ &= \frac{1}{2}(m\widehat{AD} + m\widehat{DC}) \\ &= \frac{1}{2}m\widehat{AC} \end{aligned}$$

**CASE 3** The center of the circle is not in the interior of the angle.

Let  $\angle ABC$  be an inscribed angle in which the center of the circle is not in the interior of the angle. Draw  $\overline{BOD}$ , a diameter of the circle. Then:

$$m\angle ABD = \frac{1}{2}m\widehat{AD} \quad \text{and} \quad m\angle DBC = \frac{1}{2}m\widehat{DC}$$

Therefore:

$$\begin{aligned} m\angle ABC &= m\angle ABD - m\angle DBC \\ &= \frac{1}{2}m\widehat{AD} - \frac{1}{2}m\widehat{DC} \\ &= \frac{1}{2}(m\widehat{AD} - m\widehat{DC}) \\ &= \frac{1}{2}m\widehat{AC} \end{aligned}$$

These three possible positions of the sides of the circle with respect to the center the circle prove the following theorem:

**Theorem 13.9**

The measure of an inscribed angle of a circle is equal to one-half the measure of its intercepted arc.

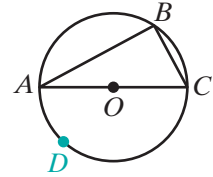
There are two statements that can be derived from this theorem.

**Corollary 13.9a**

An angle inscribed in a semicircle is a right angle.

*Proof:* In the diagram,  $\overline{AOC}$  is a diameter of circle  $O$ , and  $\angle ABC$  is inscribed in semicircle  $\widehat{ABC}$ . Also  $\widehat{ADC}$  is a semicircle whose degree measure is  $180^\circ$ . Therefore,

$$\begin{aligned} m\angle ABC &= \frac{1}{2}m\widehat{ADC} \\ &= \frac{1}{2}(180) \\ &= 90 \end{aligned}$$

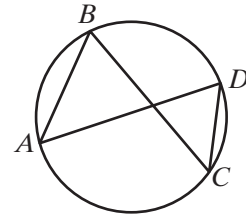


Since any triangle can be inscribed in a circle, the hypotenuse of a triangle can be the diameter of a circle with the midpoint of the hypotenuse the center of the circle.

**Corollary 13.9b**

If two inscribed angles of a circle intercept the same arc, then they are congruent.

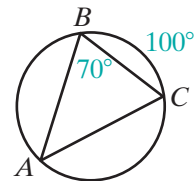
*Proof:* In the diagram,  $\angle ABC$  and  $\angle ADC$  are inscribed angles and each angle intercepts  $\widehat{AC}$ . Therefore,  $m\angle ABC = \frac{1}{2}m\widehat{AC}$  and  $m\angle ADC = \frac{1}{2}m\widehat{AC}$ . Since  $\angle ABC$  and  $\angle ADC$  have equal measures, they are congruent.

**EXAMPLE**

Triangle  $ABC$  is inscribed in circle  $O$ ,  $m\angle B = 70$ , and  $m\widehat{BC} = 100$ . Find:

- a.  $m\widehat{AC}$    b.  $m\angle A$    c.  $m\angle C$    d.  $m\widehat{AB}$

**Solution** a. If the measure of an inscribed angle is one-half the measure of its intercepted arc, then the measure of the intercepted arc is twice the measure of the inscribed angle.



$$\begin{aligned} m\angle B &= \frac{1}{2}m\widehat{AC} \\ 2m\angle B &= m\widehat{AC} \\ 2(70) &= m\widehat{AC} \\ 140 &= m\widehat{AC} \end{aligned}$$

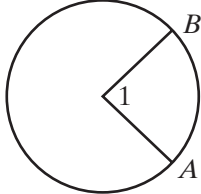
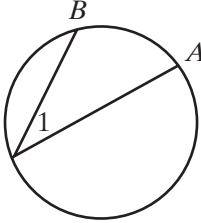
$$\begin{aligned} \text{b. } m\angle A &= \frac{1}{2}m\widehat{BC} \\ &= \frac{1}{2}(100) \\ &= 50 \end{aligned}$$

$$\begin{aligned} \text{c. } m\angle C &= 180 - (m\angle A + m\angle B) \\ &= 180 - (50 + 70) \\ &= 180 - 120 \\ &= 60 \end{aligned}$$

$$\begin{aligned} \text{d. } m\widehat{AB} &= 2m\angle C \\ &= 2(60) \\ &= 120 \end{aligned}$$

Answers a.  $140^\circ$  b.  $50^\circ$  c.  $60^\circ$  d.  $120^\circ$

### SUMMARY

Type of Angle	Degree Measure	Example
<b>Central Angle</b>	The measure of a central angle is equal to the measure of its intercepted arc.	 $m\angle 1 = m\widehat{AB}$
<b>Inscribed Angle</b>	The measure of an inscribed angle is equal to one-half the measure of its intercepted arc.	 $m\angle 1 = \frac{1}{2}m\widehat{AB}$

## Exercises

### Writing About Mathematics

1. Explain how you could use Corollary 13.9a to construct a right triangle with two given line segments as the hypotenuse and one leg.

2. In circle  $O$ ,  $\angle ABC$  is an inscribed angle and  $m\widehat{AC} = 50$ . In circle  $O'$ ,  $\angle PQR$  is an inscribed angle and  $m\widehat{PR} = 50$ . Is  $\angle ABC \cong \angle PQR$  if the circles are not congruent circles? Justify your answer.

**Developing Skills**

In 3–7,  $B$  is a point on circle  $O$  not on  $\widehat{AC}$ , an arc of circle  $O$ . Find  $m\angle ABC$  for each given  $m\widehat{AC}$ .

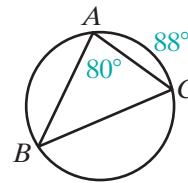
3. 88                      4. 72                      5. 170                      6. 200                      7. 280

In 8–12,  $B$  is a point on circle  $O$  not on  $\widehat{AC}$ , an arc of circle  $O$ . Find  $m\widehat{AC}$  for each given  $m\angle ABC$ .

8. 12                      9. 45                      10. 60                      11. 95                      12. 125

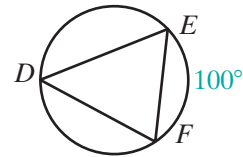
13. Triangle  $ABC$  is inscribed in a circle,  $m\angle A = 80$  and  $m\widehat{AC} = 88$ . Find:

- a.  $m\widehat{BC}$     b.  $m\angle B$     c.  $m\angle C$     d.  $m\widehat{AB}$     e.  $m\widehat{BAC}$



14. Triangle  $DEF$  is inscribed in a circle,  $\overline{DE} \cong \overline{EF}$ , and  $m\widehat{EF} = 100$ . Find:

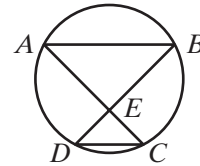
- a.  $m\angle D$     b.  $m\widehat{DE}$     c.  $m\angle F$     d.  $m\angle E$     e.  $m\widehat{DF}$



In 15–17, chords  $\overline{AC}$  and  $\overline{BD}$  intersect at  $E$  in circle  $O$ .

15. If  $m\angle B = 42$  and  $m\angle AEB = 104$ , find:

- a.  $m\angle A$     b.  $m\widehat{BC}$     c.  $m\widehat{AD}$     d.  $m\angle D$     e.  $m\angle C$



16. If  $\overline{AB} \parallel \overline{DC}$  and  $m\angle B = 40$ , find:

- a.  $m\angle D$     b.  $m\widehat{AD}$     c.  $m\widehat{BC}$     d.  $m\angle A$     e.  $m\angle DEC$

17. If  $m\widehat{AD} = 100$ ,  $m\widehat{AB} = 110$ , and  $m\widehat{BC} = 96$ , find:

- a.  $m\widehat{DC}$     b.  $m\angle A$     c.  $m\angle B$     d.  $m\angle AEB$     e.  $m\angle C$

18. Triangle  $ABC$  is inscribed in a circle and  $m\widehat{AB} : m\widehat{BC} : m\widehat{CA} = 2 : 3 : 7$ . Find:

- a.  $m\widehat{AB}$     b.  $m\widehat{BC}$     c.  $m\widehat{CA}$     d.  $m\angle A$     e.  $m\angle B$     f.  $m\angle C$

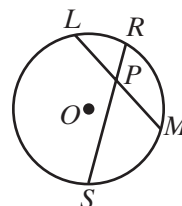
19. Triangle  $RST$  is inscribed in a circle and  $m\widehat{RS} = m\widehat{ST} = m\widehat{TR}$ . Find:

- a.  $m\widehat{RS}$       b.  $m\widehat{ST}$       c.  $m\widehat{TR}$       d.  $m\angle R$       e.  $m\angle S$       f.  $m\angle T$

### Applying Skills

20. In circle  $O$ ,  $\overline{LM}$  and  $\overline{RS}$  intersect at  $P$ .

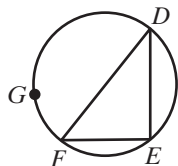
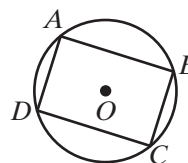
- a. Prove that  $\triangle LPR \sim \triangle SPM$ .  
 b. If  $LP = 15$  cm,  $RP = 12$  cm, and  $SP = 10$  cm, find  $MP$ .



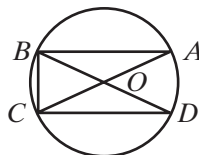
21. Triangle  $ABC$  is inscribed in a circle. If  $m\widehat{AB} = 100$  and  $m\widehat{BC} = 130$ , prove that  $\triangle ABC$  is isosceles.

22. Parallelogram  $ABCD$  is inscribed in a circle.

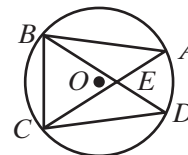
- a. Explain why  $m\widehat{ABC} = m\widehat{ADC}$ .  
 b. Find  $m\widehat{ABC}$  and  $m\widehat{ADC}$ .  
 c. Explain why parallelogram  $ABCD$  must be a rectangle.



Ex. 23



Ex. 24



Ex. 25

23. Triangle  $DEF$  is inscribed in a circle and  $G$  is any point not on  $\widehat{DEF}$ . If  $m\widehat{DE} + m\widehat{EF} = m\widehat{FGD}$ , show that  $\triangle DEF$  is a right triangle.

24. In circle  $O$ ,  $\overline{AOC}$  and  $\overline{BOD}$  are diameters. If  $\overline{AB} \cong \overline{CD}$ , prove that  $\triangle ABC \cong \triangle DCB$  by ASA.

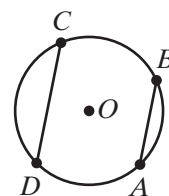
25. Chords  $\overline{AC}$  and  $\overline{BD}$  of circle  $O$  intersect at  $E$ . If  $\overline{AB} \cong \overline{CD}$ , prove that  $\triangle ABC \cong \triangle DCB$ .

26. Prove that a trapezoid inscribed in a circle is isosceles.

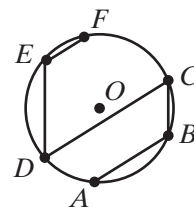
27. Minor  $\widehat{ABC}$  and major  $\widehat{ADC}$  are arcs of circle  $O$  and  $\overline{AB} \parallel \overline{CD}$ . Prove that  $\widehat{AD} \cong \widehat{BC}$ .

28. Minor  $\widehat{ABC}$  and major  $\widehat{ADC}$  are arcs of circle  $O$  and  $\widehat{AD} \cong \widehat{BC}$ . Prove that  $\overline{AB} \parallel \overline{CD}$ .

29. In circle  $O$ ,  $\overline{AOC}$  and  $\overline{BOD}$  are diameters. Prove that  $\overline{AB} \parallel \overline{CD}$ .

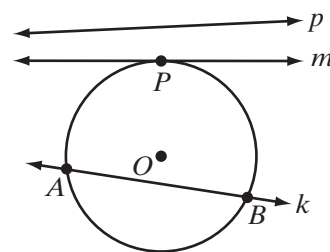


30. Points  $A, B, C, D, E,$  and  $F$  are on circle  $O$ ,  $\overline{AB} \parallel \overline{CD} \parallel \overline{EF}$ , and  $\overline{CB} \parallel \overline{ED}$ .  $ABCD$  and  $CDEF$  are trapezoids. Prove that  $\widehat{CA} \cong \widehat{BD} \cong \widehat{DF} \cong \widehat{EC}$ .
31. Quadrilateral  $ABCD$  is inscribed in circle  $O$ , and  $\widehat{AB}$  is not congruent to  $\widehat{CD}$ . Prove that  $ABCD$  is not a parallelogram.



### 13-4 TANGENTS AND SECANTS

In the diagram, line  $p$  has no points in common with the circle. Line  $m$  has one point in common with the circle. Line  $m$  is said to be *tangent to the circle*. Line  $k$  has two points in common with the circle. Line  $k$  is said to be a *secant of the circle*.



#### DEFINITION

A **tangent to a circle** is a line in the plane of the circle that intersects the circle in one and only one point.

#### DEFINITION

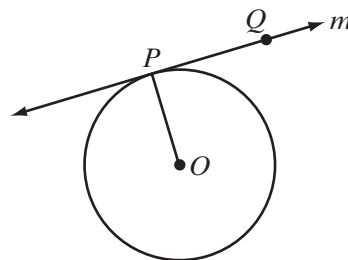
A **secant of a circle** is a line that intersects the circle in two points.

Let us begin by assuming that at every point on a circle, there exists exactly one tangent line. We can state this as a postulate.

#### Postulate 13.2

At a given point on a given circle, one and only one line can be drawn that is tangent to the circle.

Let  $P$  be any point on circle  $O$  and  $\overline{OP}$  be a radius to that point. If line  $m$  containing points  $P$  and  $Q$  is perpendicular to  $\overline{OP}$ , then  $OQ > OP$  because the perpendicular is the shortest distance from a point to a line. Therefore, every point on the line except  $P$  is outside of circle  $O$  and line  $m$  must be tangent to the circle. This establishes the truth of the following theorem.





**Theorem 13.10a**

If a line is perpendicular to a radius at a point on the circle, then the line is tangent to the circle.

The converse of this theorem is also true.

**Theorem 13.10b**

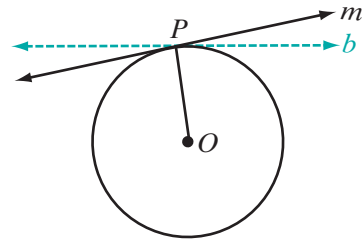
If a line is tangent to a circle, then it is perpendicular to a radius at a point on the circle.

*Given* Line  $m$  is tangent to circle  $O$  at  $P$ .

*Prove* Line  $m$  is perpendicular to  $\overline{OP}$ .

*Proof* We can use an indirect proof.

Assume that  $m$  is not perpendicular to  $\overline{OP}$ . Then there is some line  $b$  that is perpendicular to  $\overline{OP}$  at  $P$  since, at a given point on a given line, one and only one line can be drawn perpendicular to the given line. Then by Theorem 13.10a,  $b$  is a tangent to circle  $O$  at  $P$ . But this contradicts the postulate that states that at a given point on a circle, one and only one tangent can be drawn. Therefore, our assumption is false and its negation must be true. Line  $m$  is perpendicular to  $\overline{OP}$ . ■



We can state Theorems 13.10a and 13.10b as a biconditional.

**Theorem 13.10**

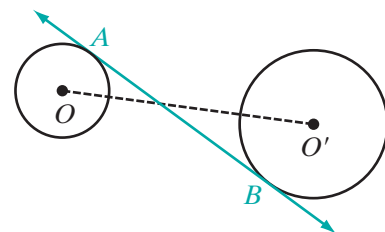
A line is tangent to a circle if and only if it is perpendicular to a radius at its point of intersection with the circle.

## Common Tangents

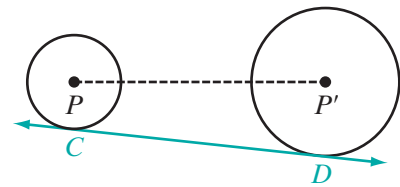
### DEFINITION

A **common tangent** is a line that is tangent to each of two circles.

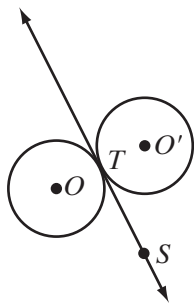
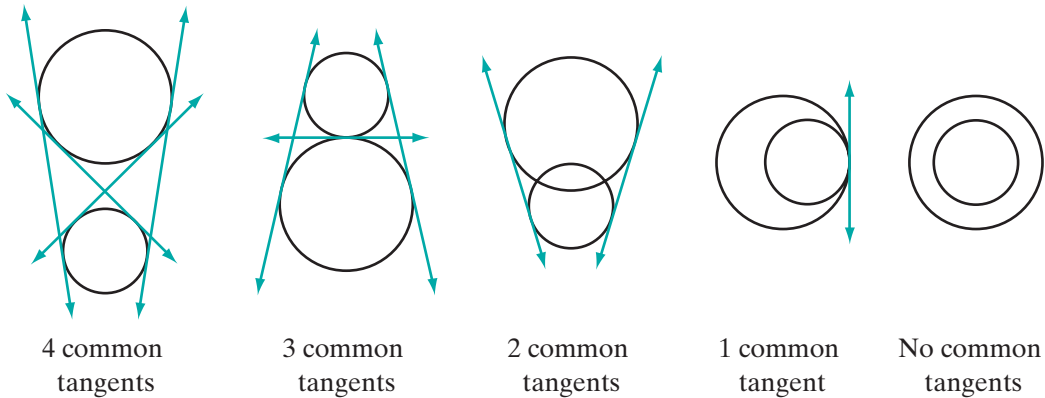
In the diagram,  $\overleftrightarrow{AB}$  is tangent to circle  $O$  at  $A$  and to circle  $O'$  at  $B$ . Tangent  $\overleftrightarrow{AB}$  is said to be a **common internal tangent** because the tangent intersects the line segment joining the centers of the circles.



In the diagram,  $\overleftrightarrow{CD}$  is tangent to circle  $P$  at  $C$  and to circle  $P'$  at  $D$ . Tangent  $\overleftrightarrow{CD}$  is said to be a **common external tangent** because the tangent does not intersect the line segment joining the centers of the circles.

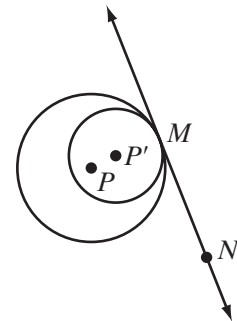


The diagrams below show that two circles can have four, three, two, one, or no common tangents.



Two circles are said to be tangent to each other if they are tangent to the same line at the same point. In the diagram,  $\overleftrightarrow{ST}$  is tangent to circle  $O$  and to circle  $O'$  at  $T$ . Circles  $O$  and  $O'$  are **tangent externally** because every point of one of the circles, except the point of tangency, is an external point of the other circle.

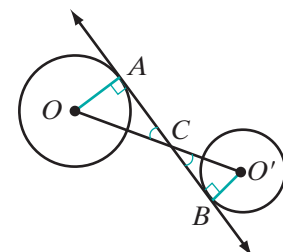
In the diagram,  $\overleftrightarrow{MN}$  is tangent to circle  $P$  and to circle  $P'$  at  $M$ . Circles  $P$  and  $P'$  are **tangent internally** because every point of one of the circles, except the point of tangency, is an internal point of the other circle.



**EXAMPLE I**

*Given:* Circles  $O$  and  $O'$  with a common internal tangent,  $\overleftrightarrow{AB}$ , tangent to circle  $O$  at  $A$  and circle  $O'$  at  $B$ , and  $C$  the intersection of  $\overline{OO'}$  and  $\overleftrightarrow{AB}$ .

*Prove:*  $\frac{AC}{BC} = \frac{OC}{O'C}$



**Proof** We will use similar triangles to prove the segments proportional.

Line  $\overleftrightarrow{AB}$  is tangent to circle  $O$  at  $A$  and circle  $O'$  at  $B$ . A line tangent to a circle is perpendicular to a radius drawn to the point of tangency. Since perpendicular lines intersect to form right angles and all right angles are congruent,  $\angle OAC \cong \angle O'BC$ . Also,  $\angle OCA \cong \angle O'CB$  because vertical angles are congruent. Therefore,  $\triangle OCA \sim \triangle O'CB$  by AA~. The lengths of corresponding sides of similar triangles are proportional. Therefore,  $\frac{AC}{BC} = \frac{OA}{O'B}$ . ■

### EXAMPLE 2

Circle  $O$  is tangent to  $\overleftrightarrow{AB}$  at  $A$ ,  $O'$  is tangent to  $\overleftrightarrow{AB}$  at  $B$ , and  $\overline{OO'}$  intersects  $\overleftrightarrow{AB}$  at  $C$ .

- Prove that  $\frac{AC}{BC} = \frac{OA}{O'B}$ .
- If  $AC = 8$ ,  $AB = 12$ , and  $OA = 9$ , find  $O'B$ .

**Solution** a. We know that  $\angle OAB \cong \angle O'BA$  because they are right angles and that  $\angle OCA \cong \angle O'CB$  because they are vertical angles. Therefore,  $\triangle OCA \sim \triangle O'CB$  by AA~ and  $\frac{AC}{BC} = \frac{OA}{O'B}$ .

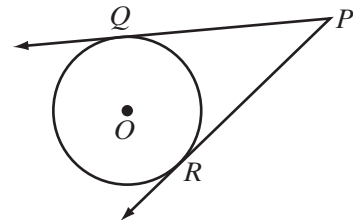
$$\begin{aligned} \text{b. } AB &= AC + BC & \frac{AC}{BC} &= \frac{OA}{O'B} \\ 12 &= 8 + BC & \frac{8}{4} &= \frac{9}{O'B} \\ 4 &= BC & 8O'B &= 36 \\ & & O'B &= \frac{36}{8} = \frac{9}{2} \text{ Answer} \end{aligned}$$
■

## Tangent Segments

### DEFINITION

A **tangent segment** is a segment of a tangent line, one of whose endpoints is the point of tangency.

In the diagram,  $\overline{PQ}$  and  $\overline{PR}$  are tangent segments of the tangents  $\overleftrightarrow{PQ}$  and  $\overleftrightarrow{PR}$  to circle  $O$  from  $P$ .



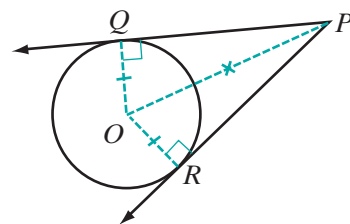
### Theorem 13.11

Tangent segments drawn to a circle from an external point are congruent.

**Given**  $\overleftrightarrow{PQ}$  tangent to circle  $O$  at  $Q$  and  $\overleftrightarrow{PR}$  tangent to circle  $O$  at  $R$ .

**Prove**  $\overline{PQ} \cong \overline{PR}$

**Proof** Draw  $\overline{OQ}$ ,  $\overline{OR}$ , and  $\overline{OP}$ . Since  $\overline{OQ}$  and  $\overline{OR}$  are both radii of the same circle,  $\overline{OQ} \cong \overline{OR}$ . Since  $\overline{QP}$  and  $\overline{RP}$  are tangent to the circle at  $Q$  and  $R$ ,  $\angle OQP$  and  $\angle ORP$  are both right angles, and  $\triangle OPQ$  and  $\triangle OPR$  are right triangles. Then  $\overline{OP}$  is the hypotenuse of both  $\triangle OPQ$  and  $\triangle OPR$ . Therefore,  $\triangle OPQ \cong \triangle OPR$  by HL. Corresponding parts of congruent triangles are congruent, so  $\overline{PQ} \cong \overline{PR}$ . ■



The following corollaries are also true.

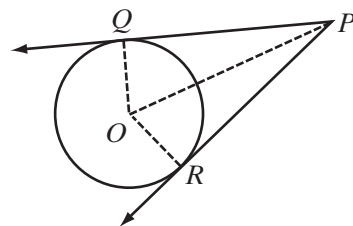
### Corollary 13.11a

If two tangents are drawn to a circle from an external point, then the line segment from the center of the circle to the external point bisects the angle formed by the tangents.

**Given**  $\overleftrightarrow{PQ}$  tangent to circle  $O$  at  $Q$  and  $\overleftrightarrow{PR}$  tangent to circle  $O$  at  $R$ .

**Prove**  $\overline{PO}$  bisects  $\angle RPQ$ .

**Strategy** Use the proof of Theorem 13.11 to show that angles  $\angle OPQ$  and  $\angle RPO$  are congruent.



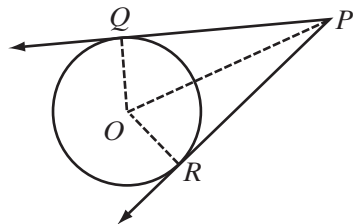
### Corollary 13.11b

If two tangents are drawn to a circle from an external point, then the line segment from the center of the circle to the external point bisects the angle whose vertex is the center of the circle and whose rays are the two radii drawn to the points of tangency.

**Given**  $\overleftrightarrow{PQ}$  tangent to circle  $O$  at  $Q$  and  $\overleftrightarrow{PR}$  tangent to circle  $O$  at  $R$ .

**Prove**  $\overline{OP}$  bisects  $\angle QOR$ .

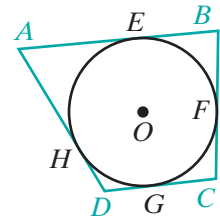
**Strategy** Use the proof of Theorem 13.11 to show that angles  $\angle QOP$  and  $\angle ROP$  are congruent.



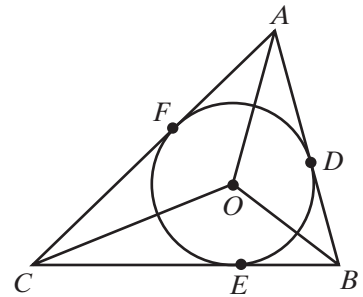
The proofs of Corollaries 13.11a and 13.11b are left to the student. (See exercises 15 and 16.)

## A Polygon Circumscribed About a Circle

A polygon is **circumscribed** about a circle if each side of the polygon is tangent to the circle. When a polygon is circumscribed about a circle, we also say that the circle is **inscribed** in the polygon. For example, in the diagram,  $\overline{AB}$  is tangent to circle  $O$  at  $E$ ,  $\overline{BC}$  is tangent to circle  $O$  at  $F$ ,  $\overline{CD}$  is tangent to circle  $O$  at  $G$ ,  $\overline{DA}$  is tangent to circle  $O$  at  $H$ . Therefore,  $ABCD$  is circumscribed about circle  $O$  and circle  $O$  is inscribed in quadrilateral  $ABCD$ .



If  $\triangle ABC$  is circumscribed about circle  $O$ , then we know that  $\overline{OA}$ ,  $\overline{OB}$ , and  $\overline{OC}$  are the bisectors of the angles of  $\triangle ABC$ , and  $O$  is the point at which the angle bisectors of the angles of a triangle intersect.



### EXAMPLE 3

$\overline{AB}$ ,  $\overline{BC}$ , and  $\overline{CA}$  are tangent to circle  $O$  at  $D$ ,  $E$ , and  $F$ , respectively. If  $AF = 6$ ,  $BE = 7$ , and  $CE = 5$ , find the perimeter of  $\triangle ABC$ .

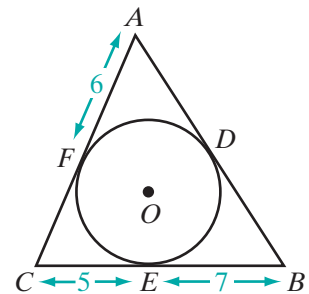
**Solution** Tangent segments drawn to a circle from an external point are congruent.

$$AD = AF = 6 \quad BD = BE = 7 \quad CF = CE = 5$$

Therefore,

$$\begin{aligned} AB &= AD + BD & BC &= BE + CE & CA &= CF + AF \\ &= 6 + 7 & &= 7 + 5 & &= 5 + 6 \\ &= 13 & &= 12 & &= 11 \end{aligned}$$

$$\begin{aligned} \text{Perimeter} &= AB + BC + CA \\ &= 13 + 12 + 11 \\ &= 36 \quad \text{Answer} \end{aligned}$$



## EXAMPLE 4

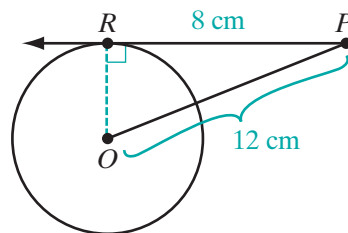
Point  $P$  is a point on a line that is tangent to circle  $O$  at  $R$ ,  $P$  is 12.0 centimeters from the center of the circle, and the length of the tangent segment from  $P$  is 8.0 centimeters.

- Find the exact length of the radius of the circle.
- Find the length of the radius to the nearest tenth.

**Solution** a.  $P$  is 12 centimeters from the center of circle  $O$ ;  $OP = 12$ .

The length of the tangent segment is 8 centimeters;  $RP = 8$ .

A line tangent to a circle is perpendicular to the radius drawn to the point of tangency;  $\triangle OPR$  is a right triangle.



$$RP^2 + OR^2 = OP^2$$

$$8^2 + OP^2 = 12^2$$

$$64 + OP^2 = 144$$

$$OP^2 = 80$$

$$OP = \sqrt{80} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$$

- b. Use a calculator to evaluate  $4\sqrt{5}$ .



ENTER: 4 **2nd**  $\sqrt{\phantom{x}}$  5 **ENTER**

DISPLAY: 8.94427191

To the nearest tenth,  $OP = 8.9$ .

**Answers** a.  $4\sqrt{5}$  cm b. 8.9 cm

## Exercises

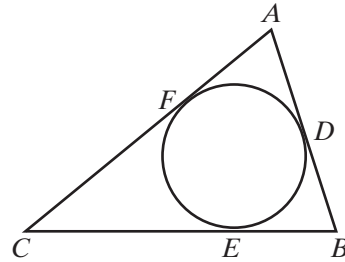
## Writing About Mathematics

- Line  $l$  is tangent to circle  $O$  at  $A$  and line  $m$  is tangent to circle  $O$  at  $B$ . If  $\overline{AOB}$  is a diameter, does  $l$  intersect  $m$ ? Justify your answer.
- Explain the difference between a polygon inscribed in a circle and a circle inscribed in a polygon.

**Developing Skills**

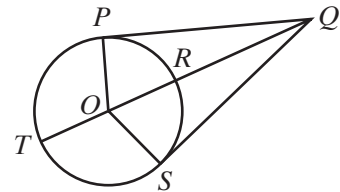
In 3 and 4,  $\triangle ABC$  is circumscribed about circle  $O$  and  $D$ ,  $E$ , and  $F$  are points of tangency.

- If  $AD = 5$ ,  $EB = 5$ , and  $CF = 10$ , find the lengths of the sides of the triangle and show that the triangle is isosceles.
- If  $AF = 10$ ,  $CE = 20$ , and  $BD = 30$ , find the lengths of the sides of the triangle and show that the triangle is a right triangle.

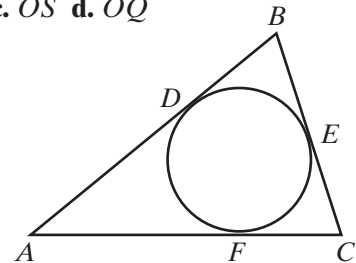


In 5–11,  $\overline{PQ}$  is tangent to circle  $O$  at  $P$ ,  $\overline{SQ}$  is tangent to circle  $O$  at  $S$ , and  $\overline{OR}$  intersects circle  $O$  at  $T$  and  $R$ .

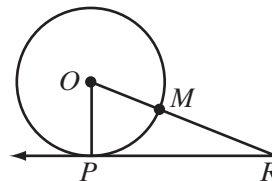
- If  $OP = 15$  and  $PQ = 20$ , find: **a.**  $OQ$  **b.**  $SQ$  **c.**  $TQ$
- If  $OQ = 25$  and  $PQ = 24$ , find: **a.**  $OP$  **b.**  $RT$  **c.**  $RQ$
- If  $OP = 10$  and  $OQ = 26$ , find: **a.**  $PQ$  **b.**  $RQ$  **c.**  $TQ$
- If  $OP = 6$  and  $TQ = 13$ , find: **a.**  $OQ$  **b.**  $PQ$  **c.**  $SQ$
- If  $OS = 9$  and  $RQ = 32$ , find: **a.**  $OQ$  **b.**  $SQ$  **c.**  $PQ$
- If  $PQ = 3x$ ,  $SQ = 5x - 8$ , and  $OS = x + 1$ , find: **a.**  $PQ$  **b.**  $SQ$  **c.**  $OS$  **d.**  $OQ$
- If  $SQ = 2x$ ,  $OS = 2x + 2$ , and  $OQ = 3x + 1$ , find: **a.**  $x$  **b.**  $SQ$  **c.**  $OS$  **d.**  $OQ$



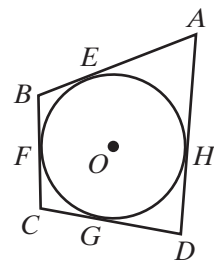
- The sides of  $\triangle ABC$  are tangent to a circle at  $D$ ,  $E$ , and  $F$ . If  $DB = 4$ ,  $BC = 7$ , and the perimeter of the triangle is 30, find: **a.**  $BE$  **b.**  $EC$  **c.**  $CF$  **d.**  $AF$  **e.**  $AC$  **f.**  $AB$



- Line  $\overleftrightarrow{RP}$  is tangent to circle  $O$  at  $P$  and  $\overline{OR}$  intersects the circle at  $M$ , the midpoint of  $\overline{OR}$ . If  $RP = 3.00$  cm, find the length of the radius of the circle: **a.** in radical form **b.** to the nearest hundredth

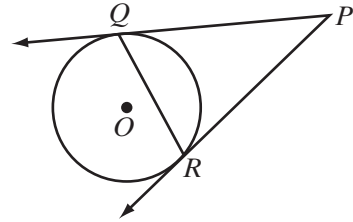


- Points  $E$ ,  $F$ ,  $G$ , and  $H$  are the points of tangency to circle  $O$  of  $\overline{AB}$ ,  $\overline{BC}$ ,  $\overline{CD}$ , and  $\overline{DA}$ , respectively. The measure of  $\widehat{EF}$  is  $80^\circ$ , of  $\widehat{FG}$  is  $70^\circ$ , and of  $\widehat{GH}$  is  $50^\circ$ . Find: **a.**  $m\angle EOF$  **b.**  $m\angle FOG$  **c.**  $m\angle GOH$  **d.**  $m\angle HOE$  **e.**  $m\angle AOE$  **f.**  $m\angle EAO$  **g.**  $m\angle EAH$  **h.**  $m\angle FBE$  **i.**  $m\angle GCF$  **j.**  $m\angle HDG$  **k.** the sum of the measures of the angles of quadrilateral  $ABCD$



Applying Skills

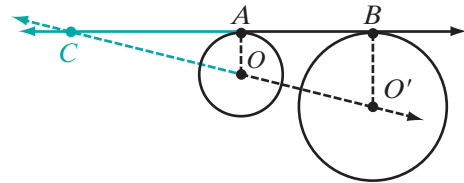
15. Prove Corollary 13.11a, “If two tangents are drawn to a circle from an external point, then the line segment from the center of the circle to the external point bisects the angle formed by the tangents.”
16. Prove Corollary 13.11b, “If two tangents are drawn to a circle from an external point, then the line segment from the center of the circle to the external point bisects the angle whose vertex is the center of the circle and whose rays are the two radii drawn to the points of tangency.”



17. Lines  $\overleftrightarrow{PQ}$  and  $\overleftrightarrow{PR}$  are tangent to circle  $O$  at  $Q$  and  $R$ .
- Prove that  $\angle PQR \cong \angle PRQ$ .
  - Draw  $\overline{OP}$  intersecting  $\overline{QR}$  at  $S$  and prove that  $QS = RS$  and  $\overline{OP} \perp \overline{QR}$ .
  - If  $OP = 10$ ,  $SQ = 4$  and  $OS < SP$ , find  $OS$  and  $SP$ .
18. Tangents  $\overline{AC}$  and  $\overline{BC}$  to circle  $O$  are perpendicular to each other at  $C$ . Prove:
- $\overline{AC} \cong \overline{AO}$
  - $OC = \sqrt{2}OA$
  - $AOBC$  is a square.

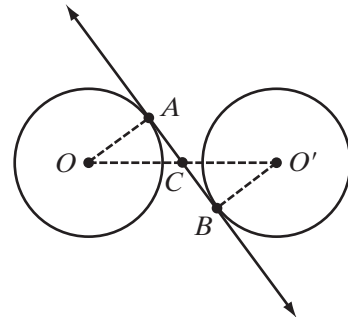
19. Isosceles  $\triangle ABC$  is circumscribed about circle  $O$ . The points of tangency of the legs,  $\overline{AB}$  and  $\overline{AC}$ , are  $D$  and  $F$ , and the point of tangency of the base,  $\overline{BC}$ , is  $E$ . Prove that  $E$  is the midpoint of  $\overline{BC}$ .

20. Line  $\overleftrightarrow{AB}$  is a common external tangent to circle  $O$  and circle  $O'$ .  $\overleftrightarrow{AB}$  is tangent to circle  $O$  at  $A$  and to circle  $O'$  at  $B$ , and  $OA < O'B$ .



- Prove that  $\overleftrightarrow{OO'}$  is not parallel to  $\overleftrightarrow{AB}$ .
- Let  $C$  be the intersection of  $\overleftrightarrow{OO'}$  and  $\overleftrightarrow{AB}$ .  
Prove that  $\triangle OAC \sim \triangle O'BC$ .
- If  $\frac{OA}{O'B} = \frac{2}{3}$ , and  $BC = 12$ , find  $AC$ ,  $AB$ ,  $OC$ ,  $O'C$ , and  $OO'$ .

21. Line  $\overleftrightarrow{AB}$  is a common internal tangent to circles  $O$  and  $O'$ .  $\overleftrightarrow{AB}$  is tangent to circle  $O$  at  $A$  and to circle  $O'$  at  $B$ , and  $OA = O'B$ . The intersection of  $\overline{OO'}$  and  $\overleftrightarrow{AB}$  is  $C$ .

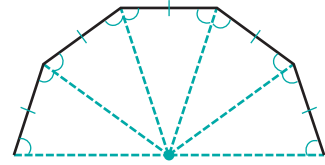


- Prove that  $OC = O'C$ .
- Prove that  $AC = BC$ .



### Hands-On Activity

Consider any regular polygon. Construct the angle bisectors of each interior angle. Since the interior angles are all congruent, the angles formed are all congruent. Since the sides of the regular polygon are all congruent, congruent isosceles triangles are formed by ASA. Any two adjacent triangles share a common leg. Therefore, they all share the same vertex. Since the legs of the triangles formed are all congruent, the vertex is equidistant from the vertices of the regular polygon. This common vertex is the **center** of the regular polygon.



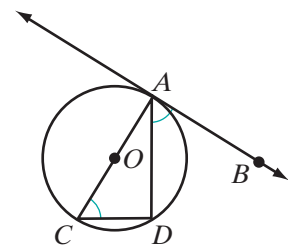
In this Hands-On Activity, we will use the center of a regular polygon to inscribe a circle in the polygon.

- a. Using geometry software or compass, protractor, and straightedge, construct a square, a regular pentagon, and a regular hexagon. For each figure:
  - (1) Construct the center  $P$  of the regular polygon. (The center is the intersection of the angle bisectors of a regular polygon.)
  - (2) Construct an **apothem** or perpendicular from  $P$  to one of the sides of regular polygon.
  - (3) Construct a circle with center  $P$  and radius equal to the length of the apothem.
- b. Prove that the circles constructed in part a are inscribed inside of the polygon. Prove:
  - (1) The apothems of each polygon are all congruent.
  - (2) The foot of each apothem is on the circle.
  - (3) The sides of the regular polygon are tangent to the circle.
- c. Let  $r$  be the distance from the center to a vertex of the regular polygon. Since the center is equidistant from each vertex, it is possible to circumscribe a circle about the polygon with radius  $r$ . Let  $a$  be the length of an apothem and  $s$  be the length of a side of the regular polygon. How is the radius,  $r$ , of the circumscribed circle related to the radius,  $a$ , of the inscribed circle?

## I 3-5 ANGLES FORMED BY TANGENTS, CHORDS, AND SECANTS

### Angles Formed by a Tangent and a Chord

In the diagram,  $\overleftrightarrow{AB}$  is tangent to circle  $O$  at  $A$ ,  $\overline{AD}$  is a chord, and  $\overline{AC}$  is a diameter. When  $\overline{CD}$  is drawn,  $\angle ADC$  is a right angle because it is an angle inscribed in a semicircle, and  $\angle ACD$  is the complement of  $\angle CAD$ . Also,  $\overline{CA} \perp \overline{AB}$ ,  $\angle BAC$  is a right angle, and  $\angle DAB$  is the complement of  $\angle CAD$ . Therefore, since complements of the same angle are congruent,  $\angle ACD \cong \angle DAB$ . We can conclude that since  $m\angle ACD = \frac{1}{2}m\widehat{AD}$ , then  $m\angle DAB = \frac{1}{2}m\widehat{AD}$ .

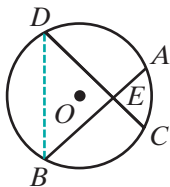


We can state what we have just proved on page 567 as a theorem.

**Theorem 13.12**

The measure of an angle formed by a tangent and a chord that intersect at the point of tangency is equal to one-half the measure of the intercepted arc.

## Angles Formed by Two Intersecting Chords



We can find how the measures of other angles and their intercepted arcs are related. For example, in the diagram, two chords  $\overline{AB}$  and  $\overline{CD}$  intersect in the interior of circle  $O$  and  $\overline{DB}$  is drawn. Angle  $AED$  is an exterior angle of  $\triangle DEB$ . Therefore,

$$\begin{aligned} m\angle AED &= m\angle BDE + m\angle DBE \\ &= \frac{1}{2}m\widehat{BC} + \frac{1}{2}m\widehat{DA} \\ &= \frac{1}{2}(m\widehat{BC} + m\widehat{DA}) \end{aligned}$$

Notice that  $\widehat{BC}$  is the arc intercepted by  $\angle BEC$  and  $\widehat{DA}$  is the arc intercepted by  $\angle AED$ , the angle vertical to  $\angle BEC$ . We can state this relationship as a theorem.

**Theorem 13.13**

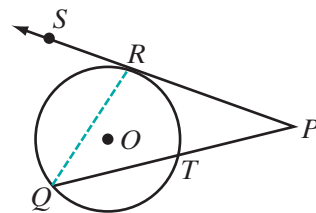
The measure of an angle formed by two chords intersecting within a circle is equal to one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

## Angles Formed by Tangents and Secants

We have shown how the measures of angles whose vertices are on the circle or within the circle are related to the measures of their intercepted arcs. Now we want to show how angles formed by two tangents, a tangent and a secant, or two secants, all of which have vertices outside the circle, are related to the measures of the intercepted arcs.

### A Tangent Intersecting a Secant

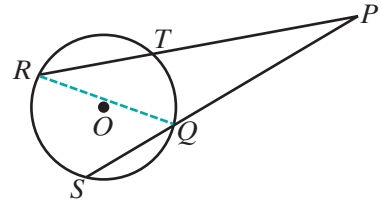
In the diagram,  $\overleftrightarrow{PRS}$  is a tangent to circle  $O$  at  $R$  and  $\overleftrightarrow{PTQ}$  is a secant that intersects the circle at  $T$  and at  $Q$ . Chord  $\overline{RQ}$  is drawn. Then  $\angle SRQ$  is an exterior angle of  $\triangle PRQ$ .



$$\begin{aligned} m\angle RQP + m\angle P &= m\angle SRQ \\ m\angle P &= m\angle SRQ - m\angle RQP \\ m\angle P &= \frac{1}{2}m\widehat{RQ} - \frac{1}{2}m\widehat{RT} \\ m\angle P &= \frac{1}{2}(m\widehat{RQ} - m\widehat{RT}) \end{aligned}$$

### Two Intersecting Secants

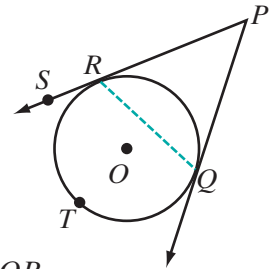
In the diagram,  $\overleftrightarrow{PTR}$  is a secant to circle  $O$  that intersects the circle at  $R$  and  $T$ , and  $\overleftrightarrow{PQS}$  is a secant to circle  $O$  that intersects the circle at  $Q$  and  $S$ . Chord  $\overline{RQ}$  is drawn. Then  $\angle RQS$  is an exterior angle of  $\triangle RQP$ .



$$\begin{aligned} m\angle PRQ + m\angle P &= m\angle RQS \\ m\angle P &= m\angle RQS - m\angle PRQ \\ m\angle P &= \frac{1}{2}m\widehat{RS} - \frac{1}{2}m\widehat{QT} \\ m\angle P &= \frac{1}{2}(m\widehat{RS} - m\widehat{QT}) \end{aligned}$$

### Two Intersecting Tangents

In the diagram,  $\overleftrightarrow{PRS}$  is tangent to circle  $O$  at  $R$ ,  $\overleftrightarrow{PQ}$  is tangent to the circle at  $Q$ , and  $T$  is a point on major  $\widehat{RQ}$ . Chord  $\overline{RQ}$  is drawn. Then  $\angle SRQ$  is an exterior angle of  $\triangle RQP$ .



$$\begin{aligned} m\angle PQR + m\angle P &= m\angle SRQ \\ m\angle P &= m\angle SRQ - m\angle PQR \\ m\angle P &= \frac{1}{2}m\widehat{RTQ} - \frac{1}{2}m\widehat{RQ} \\ m\angle P &= \frac{1}{2}(m\widehat{RTQ} - m\widehat{RQ}) \end{aligned}$$

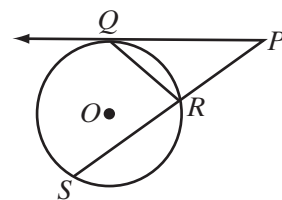
For each pair of lines, a tangent and a secant, two secants, and two tangents, the steps necessary to prove the following theorem have been given:

#### **Theorem 13.14**

The measure of an angle formed by a tangent and a secant, two secants, or two tangents intersecting outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

## EXAMPLE 1

A tangent and a secant are drawn to circle  $O$  from point  $P$ . The tangent intersects the circle at  $Q$  and the secant at  $R$  and  $S$ . If  $m\widehat{QR} : m\widehat{RS} : m\widehat{SQ} = 3 : 5 : 7$ , find:



- a.  $m\widehat{QR}$                       b.  $m\widehat{RS}$                       c.  $m\widehat{SQ}$   
 d.  $m\angle QRS$                       e.  $m\angle RQP$                       f.  $m\angle P$

**Solution** Let  $m\widehat{QR} = 3x$ ,  $m\widehat{RS} = 5x$ , and  $m\widehat{SQ} = 7x$ .

$$3x + 5x + 7x = 360$$

$$15x = 360$$

$$x = 24$$

- a.  $m\widehat{QR} = 3x$                       b.  $m\widehat{RS} = 5x$                       c.  $m\widehat{SQ} = 7x$   
        $= 3(24)$                                $= 5(24)$                                $= 7(24)$   
        $= 72$                                        $= 120$                                        $= 168$
- d.  $m\angle QRS = \frac{1}{2}m\widehat{SQ}$                       e.  $m\angle RQP = \frac{1}{2}m\widehat{QR}$                       f.  $m\angle P = \frac{1}{2}(m\widehat{SQ} - m\widehat{QR})$   
        $= \frac{1}{2}(168)$                                $= \frac{1}{2}(72)$                                $= \frac{1}{2}(168 - 72)$   
        $= 84$                                        $= 36$                                        $= 48$

**Answers** a.  $72^\circ$    b.  $120^\circ$    c.  $168^\circ$    d.  $84^\circ$    e.  $36^\circ$    f.  $48^\circ$  ■

**Note:**  $m\angle QRS = m\angle RQP + m\angle P$   
 $= 36 + 48$   
 $= 84$

## EXAMPLE 2

Two tangent segments,  $\overline{RP}$  and  $\overline{RQ}$ , are drawn to circle  $O$  from an external point  $R$ . If  $m\angle R$  is  $70$ , find the measure of the minor arc  $\widehat{PQ}$  and of the major arc  $\widehat{PSQ}$  into which the circle is divided.

**Solution** The sum of the minor arc and the major arc with the same endpoints is 360.

$$\text{Let } x = m\widehat{PQ}.$$

$$\text{Then } 360 - x = m\widehat{PSQ}.$$

$$m\angle R = \frac{1}{2}(m\widehat{PSQ} - m\widehat{PQ})$$

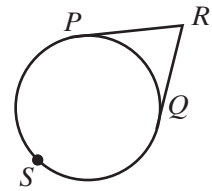
$$70 = \frac{1}{2}(360 - x - x)$$

$$70 = \frac{1}{2}(360 - 2x)$$

$$70 = 180 - x$$

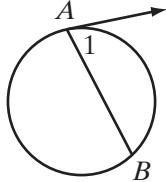
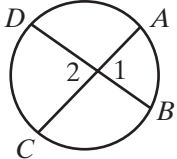
$$x = 110$$

$$\begin{aligned} 360 - x &= 360 - 110 \\ &= 250 \end{aligned}$$



**Answer**  $m\widehat{PQ} = 110$  and  $m\widehat{PSQ} = 250$  ■

### SUMMARY

Type of Angle	Degree Measure	Example
<b>Formed by a Tangent and a Chord</b>	The measure of an angle formed by a tangent and a chord that intersect at the point of tangency is equal to one-half the measure of the intercepted arc.	 $m\angle 1 = \frac{1}{2}m\widehat{AB}$
<b>Formed by Two Intersecting Chords</b>	The measure of an angle formed by two intersecting chords is equal to one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.	 $m\angle 1 = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$ $m\angle 2 = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$

(Continued)

## SUMMARY (Continued)

Type of Angle	Degree Measure	Example
Formed by Tangents and Secants	The measure of an angle formed by a tangent and a secant, two secants, or two tangents intersecting outside the circle is equal to one-half the difference of the measures of the intercepted arcs.	 $m\angle 1 = \frac{1}{2}(m\widehat{AB} - m\widehat{AC})$ $m\angle 2 = \frac{1}{2}(m\widehat{AB} - m\widehat{CD})$ $m\angle 3 = \frac{1}{2}(m\widehat{ACB} - m\widehat{AB})$

## Exercises

## Writing About Mathematics

1. Nina said that a radius drawn to the point at which a secant intersects a circle cannot be perpendicular to the secant. Do you agree with Nina? Explain why or why not.
2. Two chords intersect at the center of a circle forming four central angles. Aaron said that the measure of one of these angles is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle. Do you agree with Aaron? Explain why or why not.

## Developing Skills

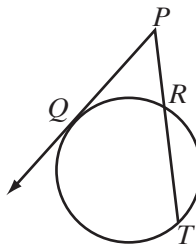
In 3–8, secants  $\overleftrightarrow{PQS}$  and  $\overleftrightarrow{PRT}$  intersect at  $P$ .

3. If  $m\widehat{ST} = 160$  and  $m\widehat{QR} = 90$ , find  $m\angle P$ .
4. If  $m\widehat{ST} = 100$  and  $m\widehat{QR} = 40$ , find  $m\angle P$ .
5. If  $m\widehat{ST} = 170$  and  $m\widehat{QR} = 110$ , find  $m\angle P$ .
6. If  $m\angle P = 40$  and  $m\widehat{QR} = 86$ , find  $m\widehat{ST}$ .
7. If  $m\angle P = 60$  and  $m\widehat{QR} = 50$ , find  $m\widehat{ST}$ .
8. If  $m\angle P = 25$  and  $m\widehat{ST} = 110$ , find  $m\widehat{QR}$ .



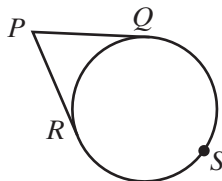
In 9–14, tangent  $\overleftrightarrow{QP}$  and secant  $\overleftrightarrow{PRT}$  intersect at  $P$ .

9. If  $m\widehat{QT} = 170$  and  $m\widehat{QR} = 70$ , find  $m\angle P$ .
10. If  $m\widehat{QT} = 120$  and  $m\widehat{QR} = 30$ , find  $m\angle P$ .
11. If  $m\widehat{QR} = 70$  and  $m\widehat{RT} = 120$ , find  $m\angle P$ .
12. If  $m\widehat{QR} = 50$  and  $m\angle P = 40$ , find  $m\widehat{QT}$ .
13. If  $m\widehat{QR} = 60$  and  $m\angle P = 35$ , find  $m\widehat{QT}$ .
14. If  $m\angle P = 30$  and  $m\widehat{QR} = 120$ , find  $m\widehat{QT}$ .



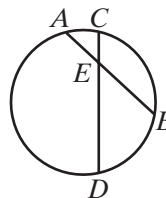
In 15–20, tangents  $\overleftrightarrow{RP}$  and  $\overleftrightarrow{QP}$  intersect at  $P$  and  $S$  is on major arc  $\widehat{QR}$ .

15. If  $m\widehat{RQ} = 160$ , find  $m\angle P$ .
16. If  $m\widehat{RQ} = 80$ , find  $m\angle P$ .
17. If  $m\widehat{RSQ} = 260$ , find  $m\angle P$ .
18. If  $m\widehat{RSQ} = 210$ , find  $m\angle P$ .
19. If  $m\widehat{RSQ} = 2m\widehat{RQ}$ , find  $m\angle P$ .
20. If  $m\angle P = 45$ , find  $m\widehat{RQ}$  and  $m\widehat{RSQ}$ .



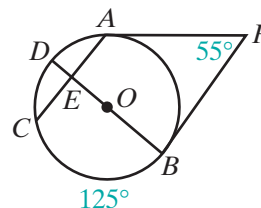
In 21–26, chords  $\overline{AB}$  and  $\overline{CD}$  intersect at  $E$  in the interior of a circle.

21. If  $m\widehat{AC} = 30$  and  $m\widehat{BD} = 80$ , find  $m\angle AEC$ .
22. If  $m\widehat{DA} = 180$  and  $m\widehat{BC} = 100$  find  $m\angle AED$ .
23. If  $m\widehat{AC} = 25$  and  $m\widehat{BD} = 45$ , find  $m\angle DEB$ .
24. If  $m\widehat{AC} = 20$  and  $m\widehat{BD} = 60$ , find  $m\angle AED$ .
25. If  $m\widehat{AC} = 30$  and  $m\angle AEC = 50$ , find  $m\widehat{BD}$ .
26. If  $m\widehat{BC} = 80$  and  $m\angle AEC = 30$ , find  $m\widehat{DA}$ .



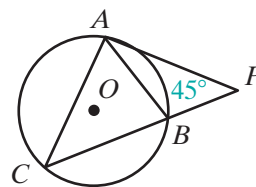
27. In the diagram,  $\overleftrightarrow{PA}$  and  $\overleftrightarrow{PB}$  are tangent to circle  $O$  at  $A$  and  $B$ . Diameter  $\overline{BD}$  and chord  $\overline{AC}$  intersect at  $E$ ,  $m\widehat{CB} = 125$  and  $m\angle P = 55$ . Find:

- a.  $m\widehat{AB}$
- b.  $m\widehat{AD}$
- c.  $m\widehat{CD}$
- d.  $m\angle DEC$
- e.  $m\angle PBD$
- f.  $m\angle PAC$
- g. Show that  $\overline{BD}$  is perpendicular to  $\overline{AC}$  and bisects  $\overline{AC}$ .



28. Tangent segment  $\overline{PA}$  and secant segment  $\overline{PBC}$  are drawn to circle  $O$  and  $\overline{AB}$  and  $\overline{AC}$  are chords. If  $m\angle P = 45$  and  $m\widehat{AC} : m\widehat{AB} = 5:2$ , find:

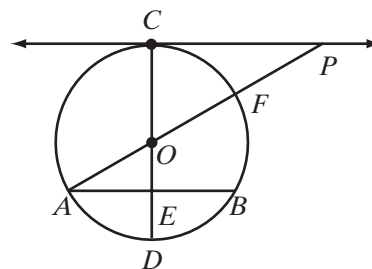
- a.  $m\widehat{AC}$       b.  $m\widehat{BC}$       c.  $m\angle ACB$   
 d.  $m\angle PAB$       e.  $m\angle CAB$       f.  $m\angle PAC$



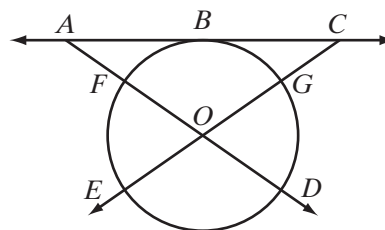
### Applying Skills

29. Tangent  $\overleftrightarrow{PC}$  intersects circle  $O$  at  $C$ , chord  $\overline{AB} \parallel \overleftrightarrow{CP}$ , diameter  $\overline{COD}$  intersects  $\overline{AB}$  at  $E$ , and diameter  $\overline{AOF}$  is extended to  $P$ .

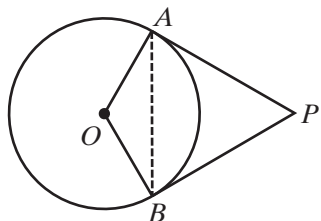
- a. Prove that  $\triangle OPC \sim \triangle OAE$ .  
 b. If  $m\angle OAE = 30$ , find  $m\widehat{AD}$ ,  $m\widehat{CF}$ ,  $m\widehat{FB}$ ,  $m\widehat{BD}$ ,  $m\widehat{AC}$ , and  $m\angle P$ .



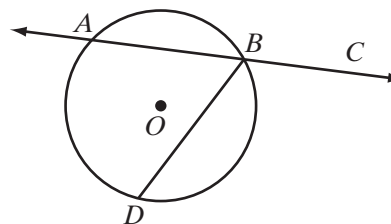
30. Tangent  $\overleftrightarrow{ABC}$  intersects circle  $O$  at  $B$ , secant  $\overleftrightarrow{AFOD}$  intersects the circle at  $F$  and  $D$ , and secant  $\overleftrightarrow{CGOE}$  intersects the circle at  $G$  and  $E$ . If  $m\widehat{EFB} = m\widehat{DGB}$ , prove that  $\triangle AOC$  is isosceles.



31. Segments  $\overline{AP}$  and  $\overline{BP}$  are tangent to circle  $O$  at  $A$  and  $B$ , respectively, and  $m\angle AOB = 120$ . Prove that  $\triangle ABP$  is equilateral.



32. Secant  $\overleftrightarrow{ABC}$  intersects a circle at  $A$  and  $B$ . Chord  $\overline{BD}$  is drawn. Prove that  $m\angle CBD \neq \frac{1}{2}m\widehat{BD}$ .





## I 3-6 MEASURES OF TANGENT SEGMENTS, CHORDS, AND SECANT SEGMENTS

### Segments Formed by Two Intersecting Chords

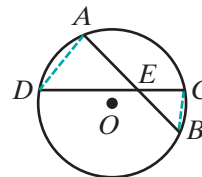
We have been proving theorems to establish the relationship between the measures of angles of a circle and the measures of the intercepted arcs. Now we will study the measures of tangent segments, secant segments, and chords. To do this, we will use what we know about similar triangles.

#### Theorem 13.15

If two chords intersect within a circle, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.

**Given** Chords  $\overline{AB}$  and  $\overline{CD}$  intersect at  $E$  in the interior of circle  $O$ .

**Prove**  $(AE)(EB) = (CE)(ED)$

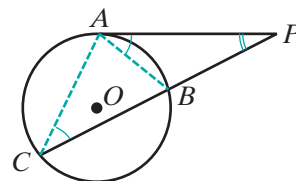


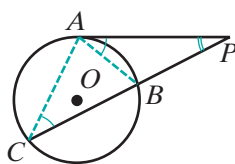
**Proof**

Statements	Reasons
1. Draw $\overline{AD}$ and $\overline{CB}$ .	1. Two points determine a line.
2. $\angle A \cong \angle C$ and $\angle D \cong \angle B$	2. Inscribed angles of a circle that intercept the same arc are congruent.
3. $\triangle ADE \sim \triangle CBE$	3. AA $\sim$ .
4. $\frac{AE}{CE} = \frac{ED}{EB}$	4. The lengths of the corresponding sides of similar triangles are in proportion.
5. $(AE)(EB) = (CE)(ED)$	5. In a proportion, the product of the means is equal to the product of the extremes. <span style="float: right;">■</span>

### Segments Formed by a Tangent Intersecting a Secant

Do similar relationships exist for tangent segments and secant segments? In the diagram, tangent segment  $\overline{PA}$  is drawn to circle  $O$ , and secant segment  $\overline{PBC}$  intersects the circle at  $B$  and  $C$ .





We will call  $\overline{PB}$ , the part of the secant segment that is outside the circle, the **external segment** of the secant. When chords  $\overline{AB}$  and  $\overline{AC}$  are drawn,  $\angle C \cong \angle PAB$  because the measure of each is one-half the measure of the intercepted arc,  $\widehat{AB}$ . Also  $\angle P \cong \angle P$  by the reflexive property. Therefore,  $\triangle BPA \sim \triangle APC$  by AA~. The length of the corresponding sides of similar triangles are in proportion. Therefore:

$$\frac{PB}{PA} = \frac{PA}{PC} \quad \text{and} \quad (PA)^2 = (PC)(PB)$$

We can write what we have just proved as a theorem:

### Theorem 13.16

If a tangent and a secant are drawn to a circle from an external point, then the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external segment.

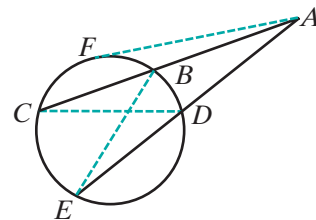
Note that both means of the proportion are  $PA$ , the length of the tangent segment. Therefore, we can say that the length of the tangent segment is the mean proportional between the lengths of the secant and its external segment. Theorem 13.16 can be stated in another way.

### Theorem 13.16

If a tangent and a secant are drawn to a circle from an external point, then the length of the tangent segment is the mean proportional between the lengths of the secant segment and its external segment.

## Segments Formed by Intersecting Secants

What is the relationship of the lengths of two secants drawn to a circle from an external point? Let  $\overline{ABC}$  and  $\overline{ADE}$  be two secant segments drawn to a circle as shown in the diagram. Draw  $\overline{AF}$  a tangent segment to the circle from  $A$ . Since



$$AF^2 = (AC)(AB) \quad \text{and} \quad AF^2 = (AE)(AD),$$

then

$$(AC)(AB) = (AE)(AD)$$

**Note:** This relationship could also have been proved by showing that  $\triangle ABE \sim \triangle ADC$ .

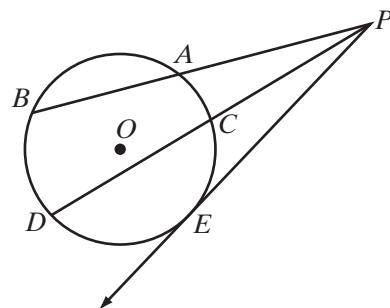
We can state this as a theorem:

**Theorem 13.17**

If two secant segments are drawn to a circle from an external point, then the product of the lengths of one secant segment and its external segment is equal to the product of the lengths of the other secant segment and its external segment.

**EXAMPLE 1**

Two secant segments,  $\overline{PAB}$  and  $\overline{PCD}$ , and a tangent segment,  $\overline{PE}$ , are drawn to a circle from an external point  $P$ . If  $PB = 9$  cm,  $PD = 12$  cm, and the external segment of  $\overline{PAB}$  is 1 centimeter longer than the external segment of  $\overline{PCD}$ , find: **a.**  $PA$    **b.**  $PC$    **c.**  $PE$



**Solution** Let  $x = PC$  and  $x + 1 = PA$ .

$$(PB)(PA) = (PD)(PC)$$

$$9PA = 12PC$$

$$9(x + 1) = 12x$$

$$9x + 9 = 12x$$

$$9 = 3x$$

$$3 = x$$

**a.**  $PA = x + 1 = 4$

**b.**  $PC = x = 3$

**c.**  $(PE)^2 = (PB)(PA)$

$$(PE)^2 = (9)(4)$$

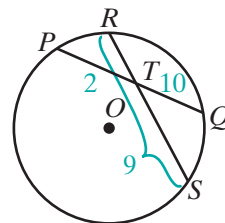
$$(PE)^2 = 36$$

$$PE = 6 \text{ (Use the positive square root.)}$$

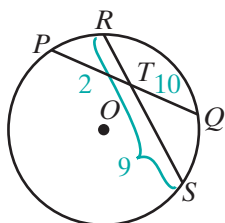
**Answers** **a.**  $PA = 4$  cm   **b.**  $PC = 3$  cm   **c.**  $PE = 6$  cm

**EXAMPLE 2**

In a circle, chords  $\overline{PQ}$  and  $\overline{RS}$  intersect at  $T$ . If  $PT = 2$ ,  $TQ = 10$ , and  $SR = 9$ , find  $ST$  and  $TR$ .



**Solution** Since  $ST + TR = SR = 9$ , let  $ST = x$  and  $TR = 9 - x$ .



$$(PT)(TQ) = (RT)(TS)$$

$$(2)(10) = x(9 - x)$$

$$20 = 9x - x^2$$

$$x^2 - 9x + 20 = 0$$

$$(x - 5)(x - 4) = 0$$

$$x - 5 = 0 \quad \Bigg| \quad x - 4 = 0$$

$$x = 5 \quad \Bigg| \quad x = 4$$

$$9 - x = 4 \quad \Bigg| \quad 9 - x = 5$$

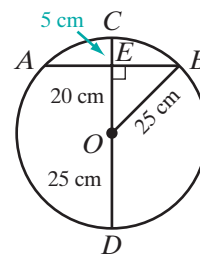
**Answer**  $ST = 5$  and  $TR = 4$ , or  $ST = 4$  and  $TR = 5$  ■

### EXAMPLE 3

Find the length of a chord that is 20 centimeters from the center of a circle if the length of the radius of the circle is 25 centimeters.

**Solution** Draw diameter  $\overline{COD}$  perpendicular to chord  $\overline{AB}$  at  $E$ . Then  $OE$  is the distance from the center of the circle to the chord.

$$\begin{aligned} OE = 20 & & DE = OD + OE & & CE = OC - OE \\ & & = 25 + 20 & & = 25 - 20 \\ & & = 45 & & = 5 \end{aligned}$$



A diameter perpendicular to a chord bisects the chord. Therefore,  $AE = EB$ .

Let  $AE = EB = x$ .

$$(AE)(EB) = (DE)(CE)$$

$$(x)(x) = (45)(5)$$

$$x^2 = 225$$

$$x = 15 \text{ (Use the positive square root.)}$$

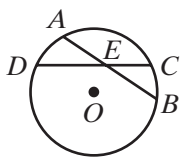
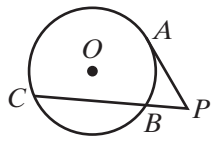
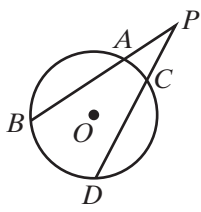
Therefore,

$$AB = AE + EB$$

$$= x + x$$

$$= 30 \text{ cm } \textbf{Answer}$$
 ■

**SUMMARY**

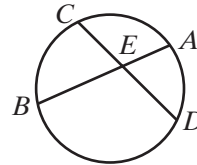
Type of Segment	Length	Example
<b>Formed by Two Intersecting Chords</b>	If two chords intersect, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.	 $(AE)(EB) = (CE)(ED)$
<b>Formed by a Tangent Intersecting a Secant</b>	If a tangent and a secant are drawn to a circle from an external point, then the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external segment.	 $(PA)^2 = (PC)(PB)$
<b>Formed by Two Intersecting Secants</b>	If two secant segments are drawn to a circle from an external point, then the product of the lengths of one secant segment and its external segment is equal to the product of the lengths of the other secant segment and its external segment.	 $(PB)(PA) = (PD)(PC)$

**Exercises**
**Writing About Mathematics**

- The length of chord  $\overline{AB}$  in circle  $O$  is 24. Vanessa said that any chord of circle  $O$  that intersects  $\overline{AB}$  at its midpoint,  $M$ , is separated by  $M$  into two segments such that the product of the lengths of the segments is 144. Do you agree with Vanessa? Justify your answer.
- Secants  $\overline{ABP}$  and  $\overline{CDP}$  are drawn to circle  $O$ . If  $AP > CP$ , is  $BP > DP$ ? Justify your answer.

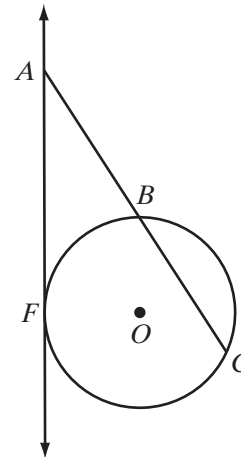
**Developing Skills**

In 3–14, chords  $\overline{AB}$  and  $\overline{CD}$  intersect at  $E$ .



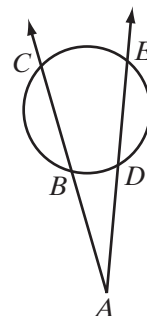
3. If  $CE = 12$ ,  $ED = 2$ , and  $AE = 3$ , find  $EB$ .
4. If  $CE = 16$ ,  $ED = 3$ , and  $AE = 8$ , find  $EB$ .
5. If  $AE = 20$ ,  $EB = 5$ , and  $CE = 10$ , find  $ED$ .
6. If  $AE = 14$ ,  $EB = 3$ , and  $ED = 6$ , find  $CE$ .
7. If  $CE = 10$ ,  $ED = 4$ , and  $AE = 5$ , find  $EB$ .
8. If  $CE = 56$ ,  $ED = 14$ , and  $AE = EB$ , find  $EB$ .
9. If  $CE = 12$ ,  $ED = 2$ , and  $AE$  is 2 more than  $EB$ , find  $EB$ .
10. If  $CE = 16$ ,  $ED = 12$ , and  $AE$  is 3 times  $EB$ , find  $EB$ .
11. If  $CE = 8$ ,  $ED = 5$ , and  $AE$  is 6 more than  $EB$ , find  $EB$ .
12. If  $CE = 9$ ,  $ED = 9$ , and  $AE$  is 24 less than  $EB$ , find  $EB$ .
13. If  $CE = 24$ ,  $ED = 5$ , and  $AB = 26$ , find  $AE$  and  $EB$ .
14. If  $AE = 7$ ,  $EB = 4$ , and  $CD = 16$ , find  $CE$  and  $ED$ .

In 15–22,  $\overrightarrow{AF}$  is tangent to circle  $O$  at  $F$  and secant  $\overline{ABC}$  intersects circle  $O$  at  $B$  and  $C$ .



15. If  $AF = 8$  and  $AB = 4$ , find  $AC$ .
16. If  $AB = 3$  and  $AC = 12$ , find  $AF$ .
17. If  $AF = 6$  and  $AC = 9$ , find  $AB$ .
18. If  $AB = 4$  and  $BC = 12$ , find  $AF$ .
19. If  $AF = 12$  and  $BC$  is 3 times  $AB$ , find  $AC$ ,  $AB$ , and  $BC$ .
20. If  $AF = 10$  and  $AC$  is 4 times  $AB$ , find  $AC$ ,  $AB$ , and  $BC$ .
21. If  $AF = 8$  and  $CB = 12$ , find  $AC$ ,  $AB$ , and  $BC$ .
22. If  $AF = 15$  and  $CB = 16$ , find  $AC$ ,  $AB$ , and  $BC$ .

In 23–30, secants  $\overrightarrow{ABC}$  and  $\overrightarrow{ADE}$  intersect at  $A$  outside the circle.



23. If  $AB = 8$ ,  $AC = 25$ , and  $AD = 10$ , find  $AE$ .
24. If  $AB = 6$ ,  $AC = 18$ , and  $AD = 9$ , find  $AE$ .
25. If  $AD = 12$ ,  $AE = 20$ , and  $AB = 8$ , find  $AC$ .
26. If  $AD = 9$ ,  $AE = 21$ , and  $AC$  is 5 times  $AB$ , find  $AB$  and  $AC$ .
27. If  $AB = 3$ ,  $AD = 2$ , and  $DE = 10$ , find  $AC$ .

28. If  $AB = 4$ ,  $BC = 12$ , and  $AD = DE$ , find  $AE$ .
29. If  $AB = 2$ ,  $BC = 7$ , and  $DE = 3$ , find  $AD$  and  $AE$ .
30. If  $AB = 6$ ,  $BC = 8$ , and  $DE = 5$ , find  $AD$  and  $AE$ .
31. In a circle, diameter  $\overline{AB}$  is extended through  $B$  to  $P$  and tangent segment  $\overline{PC}$  is drawn. If  $BP = 6$  and  $PC = 9$ , what is the measure of the diameter of the circle?

### 13-7 CIRCLES IN THE COORDINATE PLANE

In the diagram, a circle with center at the origin and a radius with a length of 5 units is drawn in the coordinate plane. The points  $(5, 0)$ ,  $(0, 5)$ ,  $(-5, 0)$  and  $(0, -5)$  are points on the circle. What other points are on the circle and what is the equation of the circle?

Let  $P(x, y)$  be any other point on the circle. From  $P$ , draw a vertical line segment to the  $x$ -axis. Let this be point  $Q$ . Then  $\triangle OPQ$  is a right triangle with  $OQ = x$ ,  $PQ = y$ , and  $OP = 5$ . We can use the Pythagorean Theorem to write an equation for the circle:

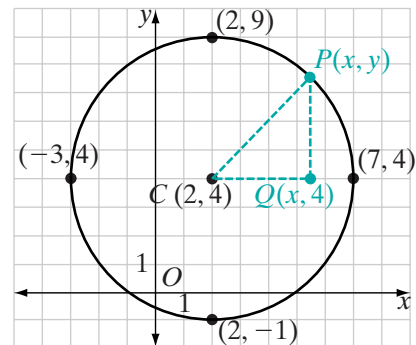
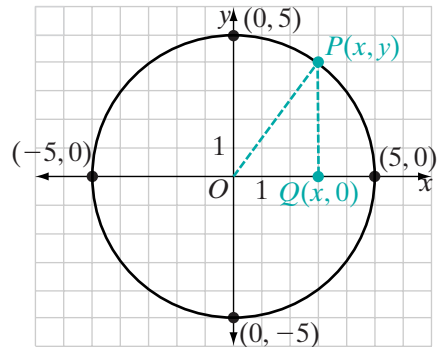
$$\begin{aligned} OQ^2 + PQ^2 &= OP^2 \\ x^2 + y^2 &= 5^2 \end{aligned}$$

The points  $(3, 4)$ ,  $(4, 3)$ ,  $(-3, 4)$ ,  $(-4, 3)$ ,  $(-3, -4)$ ,  $(-4, -3)$ ,  $(3, -4)$ , and  $(4, -3)$  appear to be points on the circle and all make the equation  $x^2 + y^2 = 5^2$  true. The points  $(5, 0)$ ,  $(0, 5)$ ,  $(-5, 0)$ , and  $(0, -5)$  also make the equation true, as do points such as  $(1, \sqrt{24})$  and  $(-2, \sqrt{21})$ .

If we replace 5 by the length of any radius,  $r$ , the equation of a circle whose center is at the origin is:

$$x^2 + y^2 = r^2$$

How does the equation change if the center is not at the origin? For example, what is the equation of a circle whose center,  $C$ , is at  $(2, 4)$  and whose radius has a length of 5 units? The points  $(7, 4)$ ,  $(-3, 4)$ ,  $(2, 9)$ , and  $(2, -1)$  are each 5 units from  $(2, 4)$  and are therefore points on the circle. Let  $P(x, y)$  be any other point on the circle. From  $P$ , draw a vertical line and from  $C$ , a horizontal line.



Let the intersection of these two lines be  $Q$ . Then  $\triangle CPQ$  is a right triangle with:

$$CQ = |x - 2| \quad PQ = |y - 4| \quad CP = 5$$

We can use the Pythagorean Theorem to write an equation for the circle.

$$\begin{aligned} CQ^2 + PQ^2 &= CP^2 \\ (x - 2)^2 + (y - 4)^2 &= 5^2 \end{aligned}$$

The points  $(5, 8)$ ,  $(6, 7)$ ,  $(-1, 8)$ ,  $(-2, 7)$ ,  $(-1, 0)$ ,  $(-2, 1)$ ,  $(5, 0)$ , and  $(6, 1)$  appear to be points on the circle and all make  $(x - 2)^2 + (y - 4)^2 = 5^2$  true. The points  $(7, 4)$ ,  $(-3, 4)$ ,  $(2, 9)$ , and  $(2, -1)$  also make the equation true, as do points whose coordinates are not integers.

We can write a general equation for a circle with center at  $C(h, k)$  and radius  $r$ . Let  $P(x, y)$  be any point on the circle. From  $P$  draw a vertical line and from  $C$  draw a horizontal line. Let the intersection of these two lines be  $Q$ . Then  $\triangle CPQ$  is a right triangle with:

$$CQ = |x - h| \quad PQ = |y - k| \quad CP = r$$

We can use the Pythagorean Theorem to write an equation for the circle.

$$\begin{aligned} CQ^2 + PQ^2 &= CP^2 \\ (x - h)^2 + (y - k)^2 &= r^2 \end{aligned}$$

In general, the **center-radius equation of a circle** with radius  $r$  and center  $(h, k)$  is

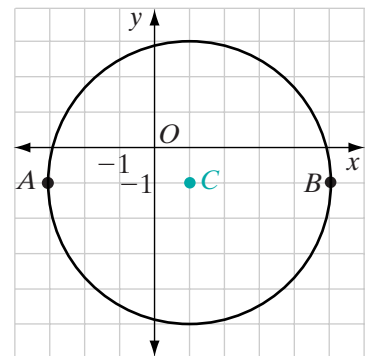
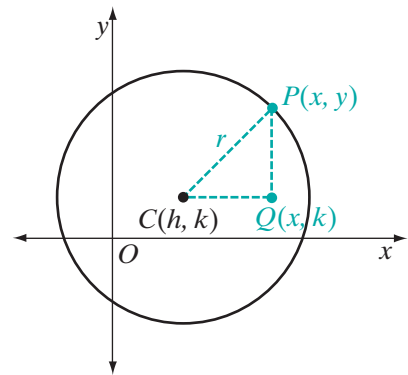
$$(x - h)^2 + (y - k)^2 = r^2$$

A circle whose diameter  $\overline{AB}$  has endpoints at  $A(-3, -1)$  and  $B(5, -1)$  is shown at the right. The center of the circle,  $C$ , is the midpoint of the diameter. Recall that the coordinates of the midpoint of the segment whose endpoints are  $(a, b)$  and  $(c, d)$  are  $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ . The coordinates of  $C$  are

$$\left(\frac{5 + (-3)}{2}, \frac{-1 + (-1)}{2}\right) = (1, -1).$$

The length of the radius is the distance from  $C$  to any point on the circle. The distance between two points on the same vertical line, that is, with the same  $x$ -coordinates, is the absolute value of the difference of the  $y$ -coordinates. The length of the radius is the distance from  $C(1, -1)$  to  $A(-3, -1)$ . The length of the radius is

$$|1 - (-3)| = 4.$$





In the equation of a circle with center at  $(h, k)$  and radius  $r$ , we use  $(x - h)^2 + (y - k)^2 = r^2$ . For this circle with center at  $(1, -1)$  and radius 4,  $h = 1$ ,  $k = -1$ , and  $r = 4$ . The equation of the circle is:

$$(x - 1)^2 + (y - (-1))^2 = 4^2 \quad \text{or} \quad (x - 1)^2 + (y + 1)^2 = 16$$

The equation of a circle is a rule for a set of ordered pairs, that is, for a relation. For the circle  $(x - 1)^2 + (y + 1)^2 = 16$ ,  $(1, -5)$  and  $(1, 3)$  are two ordered pairs of the relation. Since these two ordered pairs have the same first element, this relation is not a function.

### EXAMPLE 1

- a. Write an equation of a circle with center at  $(3, -2)$  and radius of length 7.  
 b. What are the coordinates of the endpoints of the horizontal diameter?

**Solution** a. The center of the circle is  $(h, k) = (3, -2)$ . The radius is  $r = 7$ .

The general form of the equation of a circle is  $(x - h)^2 + (y - k)^2 = r^2$ .

The equation of the given circle is:

$$(x - 3)^2 + (y - (-2))^2 = 7^2 \quad \text{or} \quad (x - 3)^2 + (y + 2)^2 = 49$$

#### b. METHOD 1

If this circle were centered at the origin, then the endpoints of the horizontal diameter would be  $(-7, 0)$  and  $(7, 0)$ . However, the circle is centered at  $(3, -2)$ . Shift these endpoints using the translation  $T_{3, -2}$ :

$$(-7, 0) \rightarrow (-7 + 3, 0 - 2) = (-4, -2)$$

$$(7, 0) \rightarrow (7 + 3, 0 - 2) = (10, -2)$$

#### METHOD 2

Since the center of the circle is  $(3, -2)$ , the  $y$ -coordinates of the endpoints are both  $-2$ . Substitute  $y = -2$  into the equation and solve for  $x$ :

$$(x - 3)^2 + (y + 2)^2 = 49$$

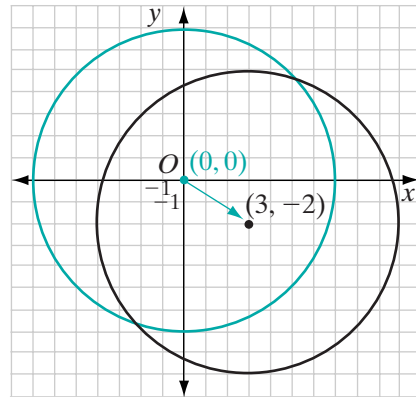
$$(x - 3)^2 + (-2 + 2)^2 = 49$$

$$x^2 - 6x + 9 + 0 = 49$$

$$x^2 - 6x - 40 = 0$$

$$(x - 10)(x + 4) = 0$$

$$x = 10 \quad | \quad x = -4$$



The coordinates of the endpoints are  $(10, -2)$  and  $(-4, -2)$ .

**Answers** a.  $(x - 3)^2 + (y + 2)^2 = 49$  b.  $(10, -2)$  and  $(-4, -2)$  ■

**EXAMPLE 2**

The equation of a circle is  $(x - 1)^2 + (y - 5)^2 = 36$ .

- What are the coordinates of the center of the circle?
- What is the length of the radius of the circle?
- What are the coordinates of two points on the circle?

**Solution** Compare the equation  $(x - 1)^2 + (y - 5)^2 = 36$  to the general form of the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

Therefore,  $h = 1$ ,  $k = 5$ ,  $r^2 = 36$ , and  $r = 6$ .

- The coordinates of the center are  $(1, 5)$ .
- The length of the radius is 6.
- Points 7 units from  $(1, 5)$  on the same horizontal line are  $(8, 5)$  and  $(-6, 5)$ .  
Points 7 units from  $(1, 5)$  on the same vertical line are  $(1, 12)$  and  $(1, -2)$ .

**Answers** a.  $(1, 5)$  b. 6 c.  $(8, 5)$  and  $(-6, 5)$  or  $(1, 12)$  and  $(1, -2)$  ■

**EXAMPLE 3**

The equation of a circle is  $x^2 + y^2 = 50$ . What is the length of the radius of the circle?

**Solution** Compare the given equation to  $x^2 + y^2 = r^2$ .

$$r^2 = 50$$

$$r = \pm\sqrt{50}$$

$$r = \pm\sqrt{25}\sqrt{2}$$

$$r = \pm 5\sqrt{2}$$

Since a length is always positive,  $r = 5\sqrt{2}$ . **Answer** ■

**Exercises****Writing About Mathematics**

- Cabel said that for every circle in the coordinate plane, there is always a diameter that is a vertical line segment and one that is a horizontal line segment. Do you agree with Cabel? Justify your answer.
- Is  $3x^2 + 3y^2 = 12$  the equation of a circle? Explain why or why not.

**Developing Skills**

In 3–8, write an equation of each circle that has the given point as center and the given value of  $r$  as the length of the radius.

3.  $(0, 0), r = 3$

4.  $(1, 3), r = 5$

5.  $(-2, 0), r = 6$

6.  $(4, -2), r = 10$

7.  $(6, 0), r = 9$

8.  $(-3, -3), r = 2$

In 9–16, write an equation of each circle that has a diameter with the given endpoints.

9.  $(-2, 0)$  and  $(2, 0)$

10.  $(0, -4)$  and  $(0, 4)$

11.  $(2, 5)$  and  $(2, 13)$

12.  $(-5, 3)$  and  $(3, 3)$

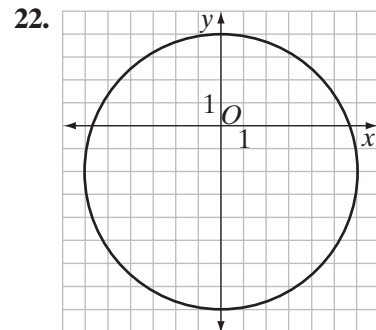
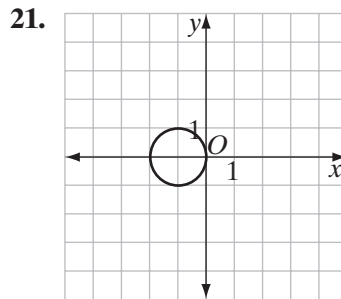
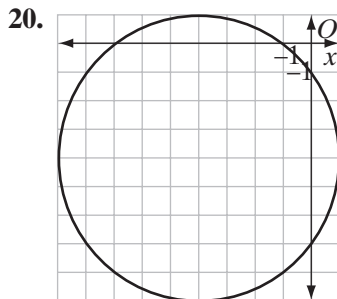
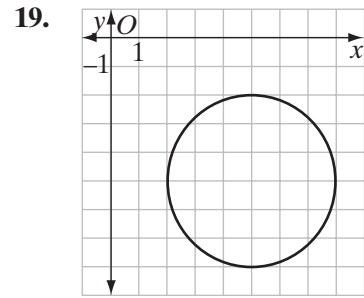
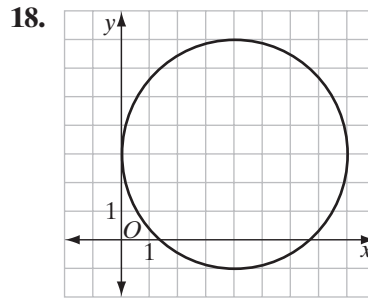
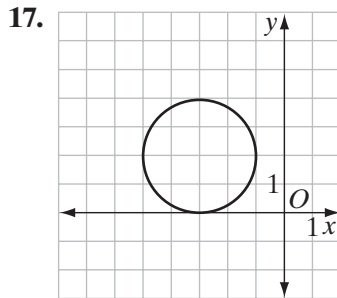
13.  $(5, 12)$  and  $(-5, 12)$

14.  $(-5, 9)$  and  $(-7, -7)$

15.  $(-7, 3)$  and  $(9, 10)$

16.  $(2, 2)$  and  $(18, -4)$

In 17–22, write an equation of each circle.



In 23–28, find the center of each circle and graph each circle.

23.  $(x - 2)^2 + (y + 5)^2 = 4$

24.  $(x + 4)^2 + (y - 4)^2 = 36$

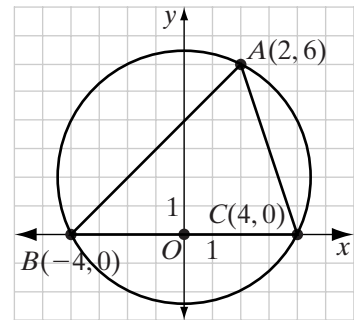
25.  $(x + \frac{3}{2})^2 + (y - 1)^2 = 25$

26.  $(x - \frac{5}{2})^2 + (y + \frac{3}{4})^2 = \frac{81}{25}$

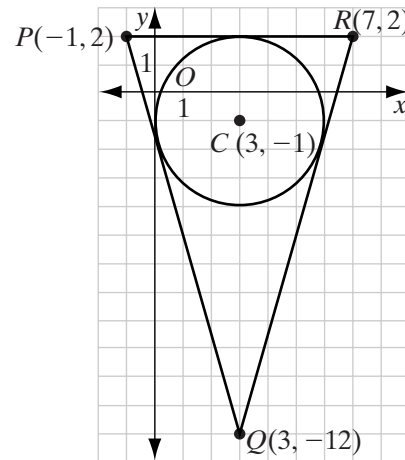
27.  $2x^2 + 2y^2 = 18$
28.  $5(x - 1)^2 + 5(y - 1)^2 = 245$
29. Point  $C(2, 3)$  is the center of a circle and  $A(-3, -9)$  is a point on the circle. Write an equation of the circle.
30. Does the point  $(4, 4)$  lie on the circle whose center is at the origin and whose radius is  $\sqrt{32}$ ? Justify your answer.
31. Is  $x^2 + 4x + 4 + y^2 - 2y + 1 = 25$  the equation of a circle? Explain why or why not.

### Applying Skills

32. In the figure on the right, the points  $A(2, 6)$ ,  $B(-4, 0)$ , and  $C(4, 0)$  appear to lie on a circle.
- Find the equation of the perpendicular bisector of  $\overline{AB}$ .
  - Find the equation of the perpendicular bisector of  $\overline{BC}$ .
  - Find the equation of the perpendicular bisector of  $\overline{AC}$ .
  - Find the circumcenter of  $\triangle ABC$ , the point of intersection of the perpendicular bisectors.
  - From what you know about perpendicular bisectors, why is the circumcenter equidistant from the vertices of  $\triangle ABC$ ?
  - Do the points  $A, B, C$  lie on a circle? Explain.

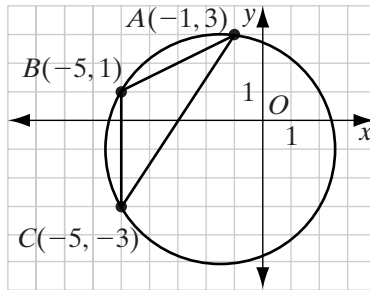


33. In the figure on the right, the circle with center at  $C(3, -1)$  appears to be inscribed in  $\triangle PQR$  with vertices  $P(-1, 2)$ ,  $Q(3, -12)$ , and  $R(7, 2)$ .



- If the equations of the angle bisectors of  $\triangle PQR$  are  $6x + 8y = 10$ ,  $x = 3$ , and  $3x - 4y = 13$ , is  $C$  the incenter of  $\triangle PQR$ ?
- From what you know about angle bisectors, why is the incenter equidistant from the sides of  $\triangle PQR$ ?
- If  $S(3, 2)$  is a point on the circle, is the circle inscribed in  $\triangle PQR$ ? Justify your answer.
- Write the equation of the circle.

34. In the figure on the right, the circle is circumscribed about  $\triangle ABC$  with vertices  $A(-1, 3)$ ,  $B(-5, 1)$ , and  $C(-5, -3)$ . Find the equation of the circle. Justify your answer algebraically.



35. Bill Bekebrede wants to build a circular pond in his garden. The garden is in the shape of an equilateral triangle. The length of the altitude to one side of the triangle is 18 feet. To plan the pond, Bill made a scale drawing on graph paper, letting one vertex of the equilateral triangle  $OAB$  be  $O(0, 0)$  and another vertex be  $A(2s, 0)$ . Therefore, the length of a side of the triangle is  $2s$ . Bill knows that an inscribed circle has its center at the intersection of the angle bisectors of the triangle. Bill also knows that the altitude, median, and angle bisector from any vertex of an equilateral triangle are the same line.
- What is the exact length, in feet, of a side of the garden?
  - In terms of  $s$ , what are the coordinates of  $B$ , the third vertex of the triangle?
  - What are the coordinates of  $C$ , the intersection of the altitudes and of the angle bisectors of the triangle?
  - What is the exact distance, in feet, from  $C$  to the sides of the garden?
  - What should be the radius of the largest possible pond?
36. The director of the town park is planning walking paths within the park. One is to be a circular path with a radius of 1,300 feet. Two straight paths are to be perpendicular to each other. One of these straight paths is to be a diameter of the circle. The other is a chord of the circle. The two straight paths intersect 800 feet from the circle. Draw a model of the paths on graph paper letting 1 unit = 100 feet. Place the center of the circle at  $(13, 13)$  and draw the diameter as a horizontal line and the chord as a vertical line.
- What is the equation of the circle?
  - What are all the possible coordinates of the points at which the straight paths intersect the circular path?
  - What are all the possible coordinates of the point at which the straight paths intersect?
  - What are the lengths of the segments into which the point of intersection separates the straight paths?

## I 3-8 TANGENTS AND SECANTS IN THE COORDINATE PLANE

### Tangents in the Coordinate Plane

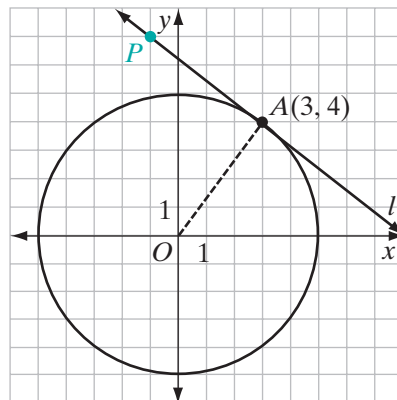
The circle with center at the origin and radius 5 is shown on the graph. Let  $l$  be a line tangent to the circle at  $A(3, 4)$ . Therefore,  $l \perp \overline{OA}$  since a tangent is perpendicular to the radius drawn to the point of tangency. The slope of  $l$  is the negative reciprocal of the slope of  $\overline{OA}$ .

$$\begin{aligned} \text{slope of } \overline{OA} &= \frac{4 - 0}{3 - 0} \\ &= \frac{4}{3} \end{aligned}$$

Therefore, the slope of  $l = -\frac{3}{4}$ . We can use the slope of  $l$  and the point  $A(3, 4)$  to write the equation of  $l$ .

$$\begin{aligned} \frac{y - 4}{x - 3} &= -\frac{3}{4} \\ 4(y - 4) &= -3(x - 3) \\ 4y - 16 &= -3x + 9 \\ 3x + 4y &= 25 \end{aligned}$$

The point  $P(-1, 7)$  makes the equation true and is therefore a point on the tangent line  $3x + 4y = 25$ .

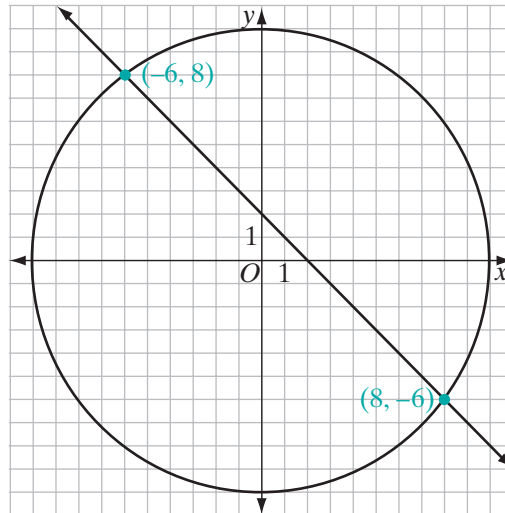


### Secants in the Coordinate Plane

A secant intersects a circle in two points. We can use an algebraic solution of a pair of equations to show that a given line is a secant. The equation of a circle with radius 10 and center at the origin is  $x^2 + y^2 = 100$ . The equation of a line in the plane is  $x + y = 2$ . Is the line a secant of the circle?

*How to Proceed*

- |  |   |
|--|---|
| (1) Solve the pair of equations algebraically:                                 | $x^2 + y^2 = 100$<br>$x + y = 2$  |
| (2) Solve the linear equation for $y$ in terms of $x$ :                        | $y = 2 - x$   |
| (3) Substitute the resulting expression for $y$ in the equation of the circle: | $x^2 + (2 - x)^2 = 100$   |
| (4) Square the binomial:   | $x^2 + 4 - 4x + x^2 = 100$  |
| (5) Write the equation in standard form:                                       | $2x^2 - 4x - 96 = 0$  |
| (6) Divide by the common factor, 2:  | $x^2 - 2x - 48 = 0$   |
| (7) Factor the quadratic equation:   | $(x - 8)(x + 6) = 0$  |
| (8) Set each factor equal to zero:   | $x - 8 = 0$ $x + 6 = 0$   |
| (9) Solve each equation for $x$ :  | $x = 8$ $x = -6$  |
| (10) For each value of $x$ find the corresponding value of $y$ :               | $y = 2 - x$ $y = 2 - x$<br>$y = 2 - 8$ $y = 2 - (-6)$<br>$y = -6$ $y = 8$ |



The common solutions are  $(8, -6)$  and  $(-6, 8)$ . The line intersects the circle in two points and is therefore a secant. In the diagram, the circle is drawn with its center at the origin and radius 10. The line  $y = 2 - x$  is drawn with a  $y$ -intercept of 2 and a slope of  $-1$ . The line intersects the circle at  $(8, -6)$  and  $(-6, 8)$ .

## EXAMPLE 1

Find the coordinates of the points at which the line  $y = 2x - 1$  intersects a circle with center at  $(0, -1)$  and radius of length  $\sqrt{20}$ .

**Solution** In the equation  $(x - h)^2 + (y - k)^2 = r^2$ , let  $h = 0$ ,  $k = -1$ , and  $r = \sqrt{20}$ . The equation of the circle is:

$$(x - 0)^2 + (y - (-1))^2 = (\sqrt{20})^2 \quad \text{or} \quad x^2 + (y + 1)^2 = 20.$$

Find the common solution of  $x^2 + (y + 1)^2 = 20$  and  $y = 2x - 1$ .

(1) The linear equation is solved for  $y$  in terms of  $x$ . Substitute, in the equation of the circle, the expression for  $y$  and simplify the result.

$$\begin{aligned} x^2 + (y + 1)^2 &= 20 \\ x^2 + (2x - 1 + 1)^2 &= 20 \\ x^2 + (2x)^2 &= 20 \end{aligned}$$

(2) Square the monomial:  $x^2 + 4x^2 = 20$

(3) Write the equation in standard form:  $5x^2 - 20 = 0$

(4) Divide by the common factor, 5:  $x^2 - 4 = 0$

(5) Factor the left side of the equation:  $(x - 2)(x + 2) = 0$

(6) Set each factor equal to zero:  $x - 2 = 0$        $x + 2 = 0$

(7) Solve each equation for  $x$ :  $x = 2$        $x = -2$

(8) For each value of  $x$  find the corresponding value of  $y$ :

$$\begin{array}{l|l} y = 2x - 1 & y = 2x - 1 \\ y = 2(2) - 1 & y = 2(-2) - 1 \\ y = 3 & y = -5 \end{array}$$

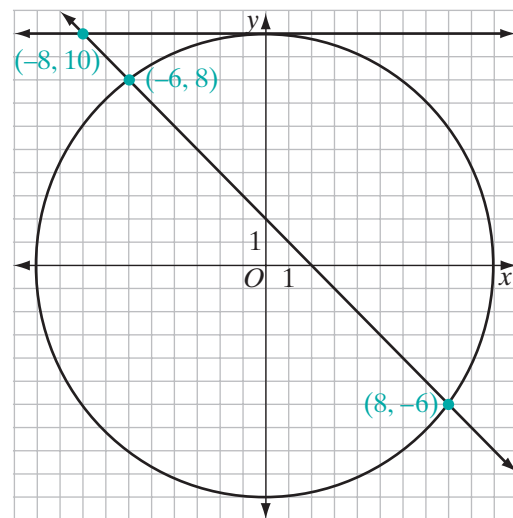
**Answer** The coordinates of the points of intersection are  $(2, 3)$  and  $(-2, -5)$ . ■

## EXAMPLE 2

The line  $x + y = 2$  intersects the circle  $x^2 + y^2 = 100$  at  $A(8, -6)$  and  $B(-6, 8)$ . The line  $y = 10$  is tangent to the circle at  $C(0, 10)$ .

- a. Find the coordinates of  $P$ , the point of intersection of the secant  $x + y = 2$  and the tangent  $y = 10$ .
- b. Show that  $PC^2 = (PA)(PB)$ .





- Solution**
- a. Use substitution to find the intersection: If  $x + y = 2$  and  $y = 10$ , then  $x + 10 = 2$  and  $x = -8$ . The coordinates of  $P$  are  $(-8, 10)$ .
- b. Use the distance formula,  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ , to find the lengths of  $PC$ ,  $PA$ , and  $PB$ .

$$\begin{aligned}
 PA &= \sqrt{(-8 - 8)^2 + (10 - (-6))^2} & PB &= \sqrt{(-8 - (-6))^2 + (10 - 8)^2} \\
 &= \sqrt{256 + 256} & &= \sqrt{4 + 4} \\
 &= \sqrt{256}\sqrt{2} & &= \sqrt{4}\sqrt{2} \\
 &= 16\sqrt{2} & &= 2\sqrt{2}
 \end{aligned}$$

$$\begin{aligned}
 PC &= \sqrt{(-8 - 0)^2 + (10 - 10)^2} \\
 &= \sqrt{64 + 0} \\
 &= 8
 \end{aligned}$$

Then:

$$\begin{aligned}
 PC^2 &= 8^2 & \text{and} & & (PA)(PB) &= (16\sqrt{2})(2\sqrt{2}) \\
 &= 64 & & & &= (32)(2) \\
 & & & & &= 64
 \end{aligned}$$

Therefore,  $PC^2 = (PA)(PB)$ . ■

## Exercises

### Writing About Mathematics

- Ron said that if the  $x$ -coordinate of the center of a circle is equal to the length of the radius of the circle, then the  $y$ -axis is tangent to the circle. Do you agree with Ron? Explain why or why not.
- At  $A$ ,  $\overleftrightarrow{AB}$  intersects a circle with center at  $C$ . The slope of  $\overleftrightarrow{AB}$  is  $m$  and the slope of  $\overline{CA}$  is  $-m$ . Is  $\overleftrightarrow{AB}$  tangent to the circle? Explain your answer.

### Developing Skills

In 3–14: **a.** Find the coordinates of the points of intersection of the circle and the line. **b.** Is the line a secant or a tangent to the circle?

- |                                      |                                      |  |
|--------------------------------------|--------------------------------------|--|
| 3. $x^2 + y^2 = 36$<br>$y = 6$       | 4. $x^2 + y^2 = 100$<br>$x + y = 14$ | 5. $x^2 + y^2 = 25$<br>$x + y = 7$       |
| 6. $x^2 + y^2 = 10$<br>$y = 3x$      | 7. $x^2 + y^2 = 9$<br>$y = x - 3$    | 8. $x^2 + y^2 = 8$<br>$x = y$            |
| 9. $x^2 + y^2 = 25$<br>$y = x - 1$   | 10. $x^2 + y^2 = 20$<br>$x + y = 6$  | 11. $x^2 + y^2 = 18$<br>$y = x + 6$      |
| 12. $x^2 + y^2 = 50$<br>$x + y = 10$ | 13. $x^2 + y^2 = 8$<br>$x + y = 4$   | 14. $x^2 + (y + 2)^2 = 4$<br>$y = x - 4$ |

In 15–18, write an equation of the line tangent to the given circle at the given point.

- $x^2 + y^2 = 9$  at  $(0, 3)$
- $x^2 + y^2 = 16$  at  $(-4, 0)$
- $x^2 + y^2 = 8$  at  $(2, -2)$
- $x^2 + y^2 = 20$  at  $(4, -2)$

### Applying Skills

- Write an equation of the secant that intersects  $x^2 + y^2 = 25$  at  $A(3, 4)$  and  $B(0, -5)$ .
  - Write an equation of the secant that intersects  $x^2 + y^2 = 25$  at  $D(0, 5)$  and  $E(0, -5)$ .
  - Find the coordinates of  $P$ , the intersection of  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{DE}$ .
  - Show that  $(PA)(PB) = (PD)(PE)$ .
- Write an equation of the secant that intersects  $x^2 + y^2 = 100$  at  $A(6, 8)$  and  $B(-8, -6)$ .
  - Write an equation of the tangent to  $x^2 + y^2 = 100$  at  $D(0, 10)$ .
  - Find the coordinates of  $P$ , the intersection of  $\overleftrightarrow{AB}$  and the tangent line at  $D$ .
  - Show that  $(PA)(PB) = (PD)^2$ .

21. a. Write an equation of the tangent to  $x^2 + y^2 = 18$  at  $A(3, 3)$ .  
 b. Write an equation of the tangent to  $x^2 + y^2 = 18$  at  $B(3, -3)$ .  
 c. Find the point  $P$  at which the tangent to  $x^2 + y^2 = 18$  at  $A$  intersects the tangent to  $x^2 + y^2 = 18$  at  $B$ .  
 d. Show that  $PA = PB$ .
22. Show that the line whose equation is  $x + 2y = 10$  is tangent to the circle whose equation is  $x^2 + y^2 = 20$ .
23. a. Show that the points  $A(-1, 7)$  and  $B(5, 7)$  lie on a circle whose radius is 5 and whose center is at  $(2, 3)$ .  
 b. What is the distance from the center of the circle to the chord  $\overline{AB}$ ?
24. Triangle  $ABC$  has vertices  $A(-7, 10)$ ,  $B(2, -2)$ , and  $C(2, 10)$ .  
 a. Find the coordinates of the points where the circle with equation  $(x + 1)^2 + (y - 7)^2 = 9$  intersects the sides of the triangle.  
 b. Show that the sides of the triangle are tangent to the circle.  
 c. Is the circle inscribed in the triangle? Explain.

## CHAPTER SUMMARY

### Definitions to Know

- A **circle** is the set of all points in a plane that are equidistant from a fixed point of the plane called the **center** of the circle.
- A **radius** of a circle (plural, *radii*) is a line segment from the center of the circle to any point of the circle.
- A **central angle of a circle** is an angle whose vertex is the center of the circle.
- An **arc of a circle** is the part of the circle between two points on the circle.
- An arc of a circle is called an **intercepted arc**, or an arc intercepted by an angle, if each endpoint of the arc is on a different ray of the angle and the other points of the arc are in the interior of the angle.
- The **degree measure of an arc** is equal to the measure of the central angle that intercepts the arc.
- **Congruent circles** are circles with congruent radii.
- **Congruent arcs** are arcs of the same circle or of congruent circles that are equal in measure.
- A **chord** of a circle is a line segment whose endpoints are points of the circle.
- A **diameter** of a circle is a chord that has the center of the circle as one of its points.
- An **inscribed angle** of a circle is an angle whose vertex is on the circle and whose sides contain chords of the circle.

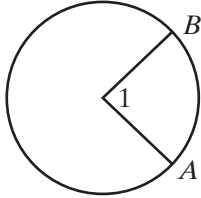
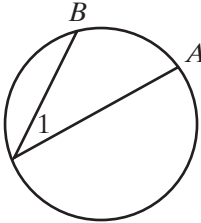
- A **tangent to a circle** is a line in the plane of the circle that intersects the circle in one and only one point.
- A **secant of a circle** is a line that intersects the circle in two points.
- A **common tangent** is a line that is tangent to each of two circles.
- A **tangent segment** is a segment of a tangent line, one of whose endpoints is the point of tangency.

- Postulates**
- 13.1** If  $\widehat{AB}$  and  $\widehat{BC}$  are two arcs of the same circle having a common endpoint and no other points in common, then  $\widehat{AB} + \widehat{BC} = \widehat{ABC}$  and  $m\widehat{AB} + m\widehat{BC} = m\widehat{ABC}$ . (**Arc Addition Postulate**)
- 13.2** At a given point on a given circle, one and only one line can be drawn that is tangent to the circle.

- Theorems and Corollaries**
- 13.1** All radii of the same circle are congruent.
- 13.2** In a circle or in congruent circles, central angles are congruent if and only if their intercepted arcs are congruent.
- 13.3** In a circle or in congruent circles, two chords are congruent if and only if their central angles are congruent.
- 13.4** In a circle or in congruent circles, two chords are congruent if and only if their arcs are congruent.
- 13.5** A diameter perpendicular to a chord bisects the chord and its arcs.
- 13.5a** A line through the center of a circle that is perpendicular to a chord bisects the chord and its arcs.
- 13.6** The perpendicular bisector of the chord of a circle contains the center of the circle.
- 13.7** Two chords are equidistant from the center of a circle if and only if the chords are congruent.
- 13.8** In a circle, if the lengths of two chords are unequal, the shorter chord is farther from the center.
- 13.9** The measure of an inscribed angle of a circle is equal to one-half the measure of its intercepted arc.
- 13.9a** An angle inscribed in a semicircle is a right angle.
- 13.9b** If two inscribed angles of a circle intercept the same arc, then they are congruent.
- 13.10** A line is tangent to a circle if and only if it is perpendicular to a radius at its point of intersection with the circle.
- 13.11** Tangent segments drawn to a circle from an external point are congruent.
- 13.11a** If two tangents are drawn to a circle from an external point, then the line segment from the center of the circle to the external point bisects the angle formed by the tangents.
- 13.11b** If two tangents are drawn to a circle from an external point, then the line segment from the center of the circle to the external point bisects the angle whose vertex is the center of the circle and whose rays are the two radii drawn to the points of tangency.

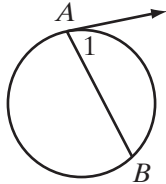
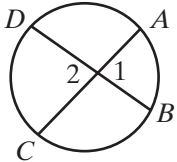
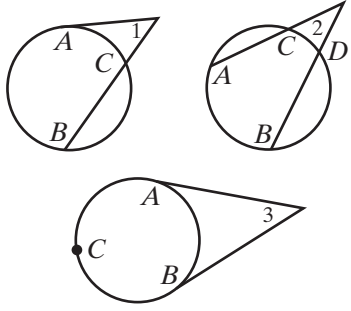
- 13.12** The measure of an angle formed by a tangent and a chord that intersect at the point of tangency is equal to one-half the measure of the intercepted arc.
- 13.13** The measure of an angle formed by two chords intersecting within a circle is equal to one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.
- 13.14** The measure of an angle formed by a tangent and a secant, two secants, or two tangents intersecting outside the circle is equal to one-half the difference of the measures of the intercepted arcs.
- 13.15** If two chords intersect within a circle, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.
- 13.16** If a tangent and a secant are drawn to a circle from an external point, then the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external segment.
- 13.16** If a tangent and a secant are drawn to a circle from an external point, then the length of the tangent segment is the mean proportional between the lengths of the secant segment and its external segment.
- 13.17** If two secant segments are drawn to a circle from an external point, then the product of the lengths of one secant segment and its external segment is equal to the product of the lengths of the other secant segment and its external segment.

### Formulas

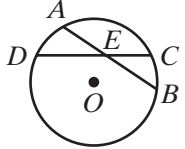
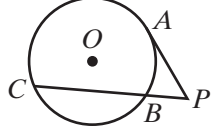
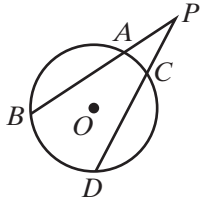
Type of Angle	Degree Measure	Example
<b>Central Angle</b>	The measure of a central angle is equal to the measure of its intercepted arc.	 $m\angle 1 = m\widehat{AB}$
<b>Inscribed Angle</b>	The measure of an inscribed angle is equal to one-half the measure of its intercepted arc.	 $m\angle 1 = \frac{1}{2}m\widehat{AB}$

(Continued)

## Formulas (Continued)

Type of Angle	Degree Measure	Example
<b>Formed by a Tangent and a Chord</b>	The measure of an angle formed by a tangent and a chord that intersect at the point of tangency is equal to one-half the measure of the intercepted arc.	 $m\angle 1 = \frac{1}{2}m\widehat{AB}$
<b>Formed by Two Intersecting Chords</b>	The measure of an angle formed by two intersecting chords is equal to one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.	 $m\angle 1 = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$ $m\angle 2 = \frac{1}{2}(m\widehat{AB} + m\widehat{CD})$
<b>Formed by Tangents and Secants</b>	The measure of an angle formed by a tangent and a secant, two secants, or two tangents intersecting outside the circle is equal to one-half the difference of the measures of the intercepted arcs.	 $m\angle 1 = \frac{1}{2}(m\widehat{AB} - m\widehat{AC})$ $m\angle 2 = \frac{1}{2}(m\widehat{AB} - m\widehat{CD})$ $m\angle 3 = \frac{1}{2}(m\widehat{ACB} - m\widehat{AB})$

**Formulas** (Continued)

Type of Segment	Length	Example
<b>Formed by Two Intersecting Chords</b>	If two chords intersect, the product of the measures of the segments of one chord is equal to the product of the measures of the segments of the other.	 $(AE)(EB) = (CE)(ED)$
<b>Formed by a Tangent Intersecting a Secant</b>	If a tangent and a secant are drawn to a circle from an external point, then the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external segment.	 $(PA)^2 = (PC)(PB)$
<b>Formed by Two Intersecting Secants</b>	If two secant segments are drawn to a circle from an external point, then the product of the lengths of one secant segment and its external segment is equal to the product of the lengths of the other secant segment and its external segment.	 $(PB)(PA) = (PD)(PC)$

The equation of a circle with radius  $r$  and center  $(h, k)$  is

$$(x - h)^2 + (y - k)^2 = r^2$$

**VOCABULARY**

**13-1** Circle • Center • Radius • Interior of a circle • Exterior of a circle • Central angle of a circle • Arc of a circle • Minor arc • Major arc • Semicircle • Intercepted arc • Degree measure of an arc • Congruent circles • Congruent arcs

**13-2** Chord • Diameter • Apothem • Inscribed polygon • Circumscribed circle

**13-3** Inscribed angle

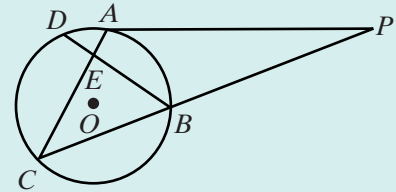
**13-4** Tangent to a circle • Secant of a circle • Common tangent • Common internal tangent • Common external tangent • Tangent externally • Tangent internally • Tangent segment • Circumscribed polygon • Inscribed circle • Center of a regular polygon

**13-6** External segment

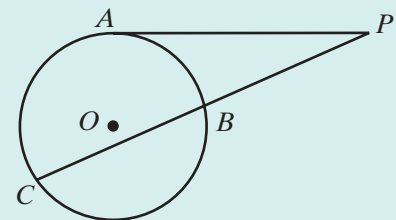
**13-7** Center-radius equation of a circle

## REVIEW EXERCISES

In 1–6,  $\overline{PA}$  is a tangent and  $\overline{PBC}$  is a secant to circle  $O$ . Chords  $\overline{AC}$  and  $\overline{BD}$  intersect at  $E$ .

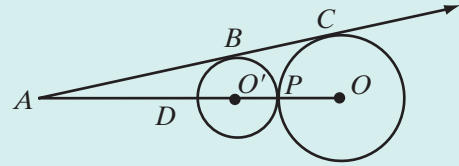


- If  $m\widehat{AB} = 80$ ,  $m\widehat{BC} = 120$ , and  $m\widehat{CD} = 100$ , find:
  - $m\angle PAC$
  - $m\angle CBD$
  - $m\angle APC$
  - $m\angle DEC$
  - $m\angle AED$
- If  $m\angle C = 50$ ,  $m\angle DBC = 55$ , and  $m\angle PAC = 100$ , find:
  - $m\widehat{AB}$
  - $m\widehat{CD}$
  - $m\angle BEC$
  - $m\angle P$
  - $m\widehat{BC}$
- If  $m\angle CEB = 80$ ,  $m\widehat{BC} = 120$ , and  $m\widehat{AB} = 70$ , find:
  - $m\widehat{AD}$
  - $m\widehat{CD}$
  - $m\angle CBD$
  - $m\angle P$
  - $m\angle PAC$
- If  $AP = 12$  and  $PC = 24$ , find  $PB$  and  $BC$ .
- If  $PB = 5$  and  $BC = 15$ , find  $AP$ .
- If  $AC = 11$ ,  $DE = 2$ ,  $EB = 12$ , and  $AE < EC$ , find  $AE$  and  $EC$ .
- Tangent segment  $\overline{PA}$  and secant segment  $\overline{PBC}$  are drawn to circle  $O$ . If  $PB = 8$  and  $BC = 10$ ,  $PA$  is equal to
  - 12
  - $4\sqrt{5}$
  - 80
  - 144
- The equation of a circle with center at  $(-2, 4)$  and radius of length 3 is
  - $(x - 2)^2 + (y - 4)^2 = 9$
  - $(x - 2)^2 + (y + 4)^2 = 9$
  - $(x + 2)^2 + (y - 4)^2 = 9$
  - $(x + 2)^2 + (y + 4)^2 = 9$
- Two tangents that intersect at  $P$  intercept a major arc of  $240^\circ$  on the circle. What is the measure of  $\angle P$ ?
- A chord that is 24 centimeters long is 9 centimeters from the center of a circle. What is the measure of the radius of the circle?





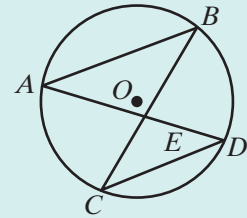
11. Two circles,  $O$  and  $O'$ , are tangent externally at  $P$ ,  $OP = 5$ , and  $O'P = 3$ . Segment  $\overline{ABC}$  is tangent to circle  $O'$  at  $B$  and to circle  $O$  at  $C$ ,  $\overline{AO'O}$  intersects circle  $O'$  at  $D$  and  $P$ , and circle  $O$  at  $P$ . If  $AD = 2$ , find  $AB$  and  $AC$ .



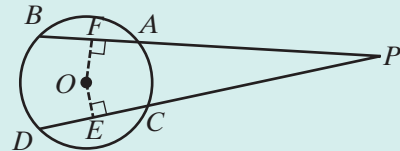
12. Isosceles  $\triangle ABC$  is inscribed in a circle. If the measure of the vertex angle,  $\angle A$ , is 20 degrees less than twice the measure of each of the base angles, find the measures of  $\widehat{AB}$ ,  $\widehat{BC}$ , and  $\widehat{CA}$ .
13. Prove that a trapezoid inscribed in a circle is isosceles.

14. In circle  $O$ , chords  $\overline{AB}$  and  $\overline{CD}$  are parallel and  $\overline{AD}$  intersects  $\overline{BC}$  at  $E$ .

- Prove that  $\triangle ABE$  and  $\triangle CDE$  are isosceles triangles.
- Prove that  $\widehat{AC} \cong \widehat{BD}$ .
- Prove that  $\triangle ABE \sim \triangle CDE$ .



15. Prove that if  $A$ ,  $B$ ,  $C$ , and  $D$  separate a circle into four congruent arcs, then quadrilateral  $ABCD$  is a square.
16. Prove, using a circumscribed circle, that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices of the triangle.
17. Secant segments  $\overline{PAB}$  and  $\overline{PCD}$  are drawn to circle  $O$ . If  $\overline{PAB} \cong \overline{PCD}$ , prove that  $\overline{AB}$  and  $\overline{CD}$  are equidistant from the center of the circle.



18. An equilateral triangle is inscribed in a circle whose radius measures 12 centimeters. How far from the center of the circle is the centroid of the triangle?

### Exploration

A regular polygon can be constructed by constructing the congruent isosceles triangles into which it can be divided. The measure of each base angle of the isosceles triangles is one-half the measure of an interior angle of the polygon. However, the interior angles of many regular polygons are not angles that can be constructed using compass and straightedge. For example, a regular polygon with nine sides has angles that measure 140 degrees. Each of the nine isosceles triangles of this polygon has base angles of 70 degrees which cannot be constructed with straightedge and compass.

In this exploration, we will construct a regular triangle (equilateral triangle), a regular hexagon, a regular quadrilateral (a square), a regular octagon, and a regular dodecagon (a polygon with 12 sides) inscribed in a circle.

- a. Explain how a compass and a straightedge can be used to construct an equilateral triangle. Prove that your construction is valid.
- b. Explain how the construction in part **a** can be used to construct a regular hexagon. Prove that your construction is valid.
- c. Explain how a square, that is, a regular quadrilateral, can be inscribed in a circle using only a compass and a straightedge. (*Hint*: What is true about the diagonals of a square?) Prove that your construction is valid.
- d. Bisect the arcs determined by the chords that are sides of the square from the construction in part **c**. Join the endpoints of the chords that are formed to draw a regular octagon. Prove that this construction is valid.
- e. A regular octagon can also be constructed by constructing eight isosceles triangles. The interior angles of a regular octagon measure 135 degrees. Bisect a right angle to construct an angle of 45 degrees. The complement of that angle is an angle of 135 degrees. Bisect this angle to construct the base angle of the isosceles triangles needed to construct a regular octagon.
- f. Explain how a regular hexagon can be inscribed in a circle using only a compass and a straightedge. (*Hint*: Recall how a regular polygon can be divided into congruent isosceles triangles.)
- g. Bisect the arcs determined by the chords that are sides of the hexagon from part **f** to draw a regular dodecagon.
- h. A regular dodecagon can also be constructed by constructing twelve isosceles triangles. The interior angles of a regular dodecagon measure 150 degrees. Bisect a 60-degree angle to construct an angle of 30 degrees. The complement of that angle is an angle of 150 degrees. Bisect this angle to construct the base angle of the isosceles triangles needed to construct a regular dodecagon.

## CUMULATIVE REVIEW

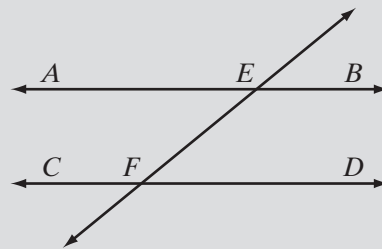
## Chapters 1–13

### Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The measure of  $\angle A$  is 12 degrees more than twice the measure of its complement. The measure of  $\angle A$  is  
(1) 26                      (2) 39                      (3) 64                      (4) 124

2. The coordinates of the midpoint of a line segment with endpoints at  $(4, 9)$  and  $(-2, 15)$  are  
 (1)  $(1, 12)$       (2)  $(3, -3)$       (3)  $(2, 24)$       (4)  $(6, -6)$
3. What is the slope of a line that is perpendicular to the line whose equation is  $2x + y = 8$ ?  
 (1)  $-2$       (2)  $2$       (3)  $-\frac{1}{2}$       (4)  $\frac{1}{2}$
4. The altitude to the hypotenuse of a right triangle separates the hypotenuse into segments of length 6 and 12. The measure of the altitude is  
 (1) 18      (2)  $3\sqrt{2}$       (3)  $6\sqrt{2}$       (4)  $6\sqrt{3}$
5. The diagonals of a quadrilateral bisect each other. The quadrilateral cannot be a  
 (1) trapezoid      (2) rectangle      (3) rhombus      (4) square
6. Two triangles,  $\triangle ABC$  and  $\triangle DEF$ , are similar. If  $AB = 12$ ,  $DE = 18$ , and the perimeter of  $\triangle ABC$  is 36, then the perimeter of  $\triangle DEF$  is  
 (1) 24      (2) 42      (3) 54      (4) 162
7. Which of the following do not always lie in the same plane?  
 (1) two points      (3) two lines  
 (2) three points      (4) a line and a point not on the line
8. At  $A$ , the measure of an exterior angle of  $\triangle ABC$  is 110 degrees. If the measure of  $\angle B$  is 45 degrees, what is the measure of  $\angle C$ ?  
 (1) 55      (2) 65      (3) 70      (4) 135
9. Under the composition  $r_{x\text{-axis}} \circ T_{2,3}$ , what are the coordinates of the image of  $A(3, -5)$ ?  
 (1)  $(5, 2)$       (2)  $(-5, -2)$       (3)  $(5, 8)$       (4)  $(-1, -2)$
10. In the diagram,  $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$  and  $\overleftrightarrow{EF}$  intersects  $\overleftrightarrow{AB}$  at  $E$  and  $\overleftrightarrow{CD}$  at  $F$ . If  $m\angle AEF$  is represented by  $3x$  and  $m\angle CFE$  is represented by  $2x + 20$ , what is the value of  $x$ ?  
 (1) 4      (3) 32  
 (2) 12      (4) 96



## Part II

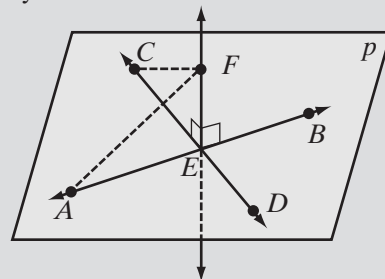
Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The coordinates of the vertices of  $\triangle ABC$  are  $A(8, 0)$ ,  $B(-4, 4)$ , and  $C(0, -4)$ .
  - a. Find the coordinates of  $D$ , the midpoint of  $\overline{AB}$ .
  - b. Find the coordinates of  $E$ , the midpoint of  $\overline{BC}$ .
  - c. Is  $\overline{DE} \parallel \overline{AC}$ ? Justify your answer.
  - d. Are  $\triangle ABC$  and  $\triangle DBE$  similar triangles? Justify your answer.
12.  $ABCD$  is a quadrilateral,  $\overline{AC} \cong \overline{BD}$  and  $\overline{AC}$  and  $\overline{BD}$  bisect each other at  $E$ . Prove that  $ABCD$  is a rectangle.

## Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. In the diagram,  $\overleftrightarrow{AB}$  and  $\overleftrightarrow{CD}$  intersect at  $E$  in plane  $p$ ,  $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$ , and  $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$ . If  $\overline{EA} \cong \overline{EC}$ , prove that  $\overline{FA} \cong \overline{FC}$ .



14. The length of the hypotenuse of a right triangle is 2 more than the length of the longer leg. The length of the shorter leg is 7 less than the length of the longer leg. Find the lengths of the sides of the right triangle.

## Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. a. Find the coordinates of  $A'$ , the image of  $A(5, 2)$  under the composition  $R_{90} \circ r_{y=x}$ .
- b. What single transformation is equivalent to  $R_{90} \circ r_{y=x}$ ?
- c. Is  $R_{90} \circ r_{y=x}$  a direct isometry? Justify your answer.

16. In the diagram,  $D$  is a point on  $\overline{ADC}$  such that  $AD : DC = 1 : 3$ , and  $E$  is a point on  $\overline{BEC}$  such that  $BE : EC = 1 : 3$ .

- a. Show that  $AC : DC = BC : EC$ .
- b. Prove that  $\triangle ABC \sim \triangle DEC$ .

