CHAPTER

6

TRANSFORMATIONS AND THE COORDINATE PLANE

In our study of mathematics, we are accustomed to representing an equation as a line or a curve. This blending of geometry and algebra was not always familiar to mathematicians. In the seventeenth century, René Descartes (1596–1650), a French philosopher and mathematician, applied algebraic principles and methods to geometry. This blending of algebra and geometry is known as **coordinate geometry** or **analytic geometry**. Pierre de Fermat (1601–1665) independently developed analytic geometry before Descartes but Descartes was the first to publish his work.

Descartes showed that a curve in a plane could be completely determined if the distances of its points from two fixed perpendicular lines were known. We call these distances the **Cartesian coordinates** of the points; thus giving Descartes' name to the system that he developed.

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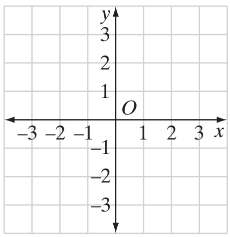
6-1 THE COORDINATES OF A POINT IN A PLANE



In this chapter, we will review what you know about the coordinate plane and you will study *transformations* and *symmetry*, which are common elements in nature, art, and architecture.

Two intersecting lines determine a plane. The **coordinate plane** is determined by a horizontal line, the ***x*-axis**, and a vertical line, the ***y*-axis**, which are perpendicular and intersect at a point called the **origin**. Every point on a plane can be described by two numbers, called the **coordinates** of the point, usually written as an **ordered pair**. The first number in the pair, called the ***x*-coordinate** or the **abscissa**, is the distance from the point to the *y*-axis. The second number, the ***y*-coordinate** or the **ordinate** is the distance from the point to the *x*-axis. In general, the coordinates of a point are represented as **(x, y)**. Point O, the origin, has the coordinates (0, 0).

We will accept the following postulates of the coordinate plane.

Postulate 6.1 Two points are on the same horizontal line if and only if they have the same *y*-coordinates.

Postulate 6.2 The length of a horizontal line segment is the absolute value of the difference of the *x*-coordinates.

Postulate 6.3 Two points are on the same vertical line if and only if they have the same *x*-coordinate.

Postulate 6.4 The length of a vertical line segment is the absolute value of the difference of the *y*-coordinates.

Postulate 6.5 Each vertical line is perpendicular to each horizontal line.

Locating a Point in the Coordinate Plane

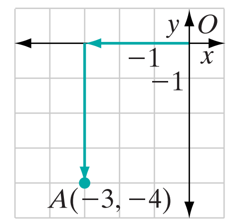
An ordered pair of signed numbers uniquely determines the location of a point in the plane.

Procedure

To locate a point in the coordinate plane:

1. From the origin, move along the *x*-axis the number of units given by the *x*-coordinate. Move to the right if the number is positive or to the left if the number is negative. If the *x*-coordinate is 0, there is no movement along the *x*-axis.
2. Then, from the point on the *x*-axis, move parallel to the *y*-axis the number of units given by the *y*-coordinate. Move up if the number is positive or down if the number is negative. If the *y*-coordinate is 0, there is no movement in the *y* direction.

For example, to locate the point $A(-3, -4)$, from O, move 3 units to the left along the x-axis, then 4 units down, parallel to the y-axis.



Finding the Coordinates of a Point in a Plane

The location of a point in the coordinate plane uniquely determines the coordinates of the point.

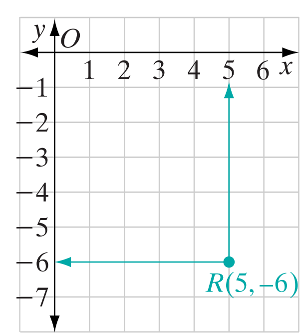
Procedure

To find the coordinates of a point:

1. From the point, move along a vertical line to the *x*-axis. The number on the *x*-axis is the *x*-coordinate of the point.
2. From the point, move along a horizontal line to the *y*-axis. The number on the *y*-axis is the *y*-coordinate of the point.

For example, from point R, move in the vertical direction to 5 on the x-axis and in the horizontal direction to -6 on the y-axis. The coordinates of R are $(5, -6)$.

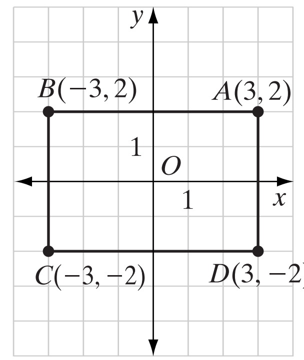
Note: The coordinates of a point are often referred to as **rectangular coordinates**.



Graphing Polygons

A quadrilateral (a four-sided polygon) can be represented in the coordinate plane by locating its vertices and then drawing the sides connecting the vertices in order.

The graph shows the quadrilateral $ABCD$. The vertices are $A(3, 2)$, $B(-3, 2)$, $C(-3, -2)$ and $D(3, -2)$.



1. Points A and B have the same y-coordinate and are on the same horizontal line.
2. Points C and D have the same y-coordinate and are on the same horizontal line.
3. Points B and C have the same x-coordinate and are on the same vertical line.
4. Points A and D have the same x-coordinate and are on the same vertical line.
5. Every vertical line is perpendicular to every horizontal line.
6. Perpendicular lines are lines that intersect to form right angles.

Each angle of the quadrilateral is a right angle:

$$m\angle A = m\angle B = m\angle C = m\angle D = 90$$

From the graph, we can find the dimensions of this quadrilateral. To find AB and CD, we can count the number of units from A to B or from C to D.

$$AB = CD = 6$$

Points on the same horizontal line have the same y-coordinate. Therefore, we can also find AB and CD by subtracting their x-coordinates.

$$\begin{aligned} AB = CD &= 3 - (-3) \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

To find BC and DA, we can count the number of units from B to C or from D to A.

$$BC = DA = 4$$

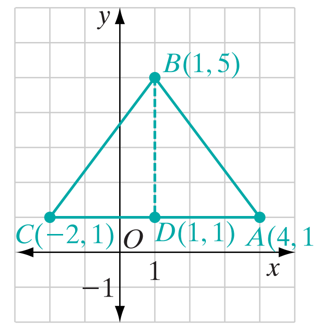
Points on the same vertical line have the same x-coordinate. Therefore, we can find BC and DA by subtracting their y-coordinates.

$$\begin{aligned} BC = DA &= 2 - (-2) \\ &= 2 + 2 \\ &= 4 \end{aligned}$$

EXAMPLE 1

Graph the following points: $A(4, 1)$, $B(1, 5)$, $C(-2,1)$. Then draw $\triangle ABC$ and find its area.

Solution The graph shows $\triangle ABC$.



To find the area of the triangle, we need to know the lengths of the base and of the altitude drawn to that base. The base of $\triangle ABC$ is $\overline{AC}$, a horizontal line segment.

$$\begin{aligned} AC &= 4 - (-2) \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

The vertical line segment drawn from B perpendicular to $\overline{AC}$ is the altitude $\overline{BD}$.

$$\begin{aligned} BD &= 5 - 1 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(AC)(BD) \\ &= \tfrac{1}{2}(6)(4) \\ &= 12 \end{aligned}$$

Answer The area of $\triangle ABC$ is 12 square units.

Exercises

Writing About Mathematics

1. Mark is drawing isosceles right triangle ABC on the coordinate plane. He locates points $A(-2, -4)$ and $C(5, -4)$. He wants the right angle to be $\angle C$. What must be the coordinates of point B? Explain how you found your answer.

2. Phyllis graphed the points $D(3, 0)$, $E(0, 5)$, $F(-2, 0)$, and $G(0, -4)$ on the coordinate plane and joined the points in order. Explain how Phyllis can find the area of this polygon.

Developing Skills

In 3–12: **a.** Graph the points and connect them with straight lines in order, forming a polygon. **b.** Find the area of the polygon.

3. $A(1, 1)$, $B(8, 1)$, $C(1, 5)$

4. $P(0, 0)$, $Q(5, 0)$, $R(5, 4)$, $S(0, 4)$

5. $C(8, -1)$, $A(9, 3)$, $L(4, 3)$, $F(3, -1)$

6. $H(-4, 0)$, $O(0, 0)$, $M(0, 4)$, $E(-4, 4)$

7. $H(5, -3), E(5, 3), N(-2, 0)$

8. $F(5, 1), A(5, 5), R(0, 5), M(-2, 1)$

9. $B(-3, -2), A(2, -2), R(2, 2), N(-3, 2)$

10. $P(-3, 0), O(0, 0), N(2, 2), D(-1, 2)$

11. $R(-4, 2), A(0, 2), M(0, 7)$

12. $M(-1, -1), I(3, -1), L(3, 3), K(-1, 3)$

13. Graph points $A(1, 1)$, $B(5, 1)$, and $C(5, 4)$. What must be the coordinates of point D if $ABCD$ is a quadrilateral with four right angles?

14. Graph points $P(-1, -4)$ and $Q(2, -4)$. What are the coordinates of R and S if $PQRS$ is a quadrilateral with four right angles and four congruent sides? (Two answers are possible.)

15. **a.** Graph points $S(3, 0)$, $T(0, 4)$, $A(-3, 0)$, and $R(0, -4)$, and draw the polygon $STAR$.

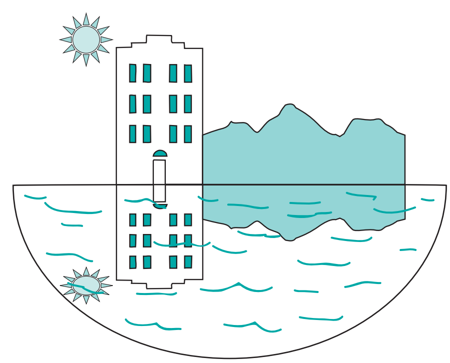
b. Find the area of $STAR$ by adding the areas of the triangles into which the axes separate the polygon.

16. **a.** Graph points $P(2, 0)$, $L(1, 1)$, $A(-1, 1)$, $N(-2, 0)$. $E(-1, -1)$, and $T(1, -1)$, and draw the hexagon $PLANET$.

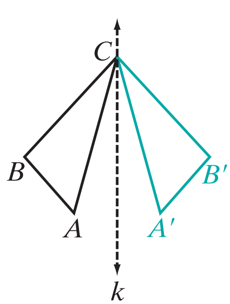
b. Find the area of $PLANET$. (*Hint:* Use two vertical lines to separate the hexagon into parts.)

6-2 LINE REFLECTIONS

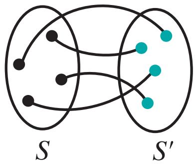
It is often possible to see the objects along the shore of a body of water reflected in the water. If a picture of such a scene is folded, the objects can be made to coincide with their images. Each point of the reflection is an image point of the corresponding point of the object.



The line along which the picture is folded is the **line of reflection**, and the correspondence between the object points and the image points is called a **line reflection**. This common experience is used in mathematics to study congruent figures.



If the figure at the left were folded along line k, $\triangle ABC$ would coincide with $\triangle A'B'C$. Line k is the line of reflection, point A corresponds to point A' (in symbols, $A \to A'$) and point B corresponds to point B' ($B \to B'$). Point C is a **fixed point** because it is a point on the line of reflection. In other words, C corresponds to itself ($C \to C$). Under a reflection in line k, then, $\triangle ABC$ corresponds to $\triangle A'B'C$ ($\triangle ABC \to \triangle A'B'C$). Each of the points A, B, and C is called a *preimage* and each of the points A', B', and C is called an *image*.



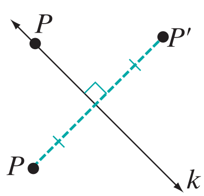
DEFINITION

A **transformation** is a one-to-one correspondence between two sets of points, S and S', such that every point in set S corresponds to one and only one point in set S', called its **image**, and every point in S' is the image of one and only one point in S, called its **preimage**.

The sets S and S' can be the same set and that set of points is frequently the set of points in a plane. For example, let S be the set of points in the coordinate plane. Let the image of (x, y), a point in S, be $(2 - x, y)$, a point in S'. Under this transformation:

$$(0, 1) \rightarrow (2 - 0, 1) \quad = (2, 1)$$
$$(-4, 3) \rightarrow (2 - (-4), 3) = (-2, 3)$$
$$(5, -1) \rightarrow (2 - 5, -1) = (-3, -1)$$

Every image $(2 - x, y)$ in S' is a point of the coordinate plane, that is, a point of S. Therefore, $S' = S$.



DEFINITION

A **reflection in line *k*** is a transformation in a plane such that:

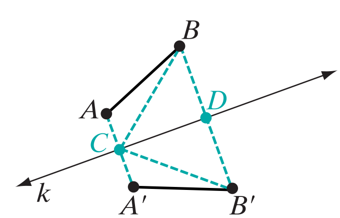
1. If point P is not on k, then the image of P is P' where k is the perpendicular bisector of $\overline{PP'}$.
2. If point P is on k, the image of P is P.

From the examples that we have seen, it appears that the size and shape of the image is the same as the size and shape of the preimage. We want to prove that this is true.

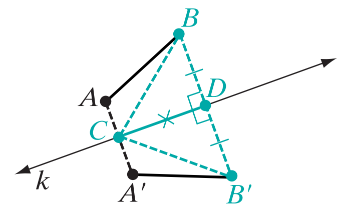
Theorem 6.1 Under a line reflection, distance is preserved.

Given Under a reflection in line k, the image of A is A' and the image of B is B'.

Prove $AB = A'B'$

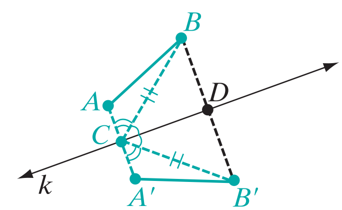


Proof We will prove this theorem by using the definition of a reflection and SAS to show that $\triangle BDC \cong \triangle B'DC$ and that $\overline{BC} \cong \overline{B'C}$. Then, using the fact that if two angles are congruent, their complements are congruent (Theorem 4.4), we can show that $\triangle ACB \cong \triangle A'CB'$. From this last statement, we can conclude that $AB = A'B'$.



(1) Let the image of A be A' and the image of B be B' under reflection in k. Let C be the midpoint of $\overline{AA'}$ and D be the midpoint of $\overline{BB'}$. Points C and D are on k, since k is the perpendicular bisector of $\overline{AA'}$ and of $\overline{BB'}$.

(2) $\overline{BD} \cong \overline{B'D}$ since D is the midpoint of $\overline{BB'}$, $\angle BDC \cong \angle B'DC$ since perpendicular lines intersect to form right angles, and $\overline{CD} \cong \overline{CD}$ by the reflexive property. Thus, $\triangle BDC \cong \triangle B'DC$ by SAS.



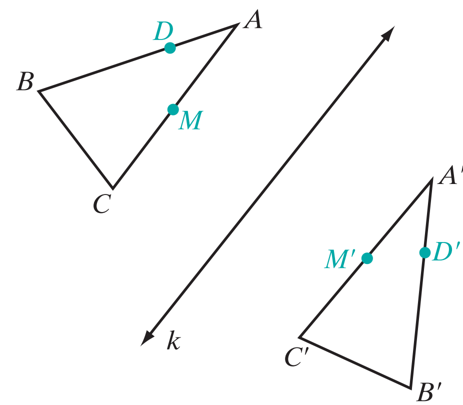
(3) From step 2, we can conclude that $\overline{BC} \cong \overline{B'C}$ and $\angle BCD \cong \angle B'CD$ since corresponding parts of congruent triangles are congruent.

(4) Since k is the perpendicular bisector of $\overline{AA'}$, $\angle ACD$ and $\angle A'CD$ are right angles. Thus, $\angle ACB$ and $\angle BCD$ are complementary angles. Similarly, $\angle A'CB'$ and $\angle B'CD$ are complementary angles. If two angles are congruent, their complements are congruent. Since $\angle BCD$ and $\angle B'CD$ were shown to be congruent in step 3, their complements are congruent: $\angle ACB \cong \angle A'CB'$.

(5) By SAS, $\triangle ACB \cong \triangle A'CB'$. Since corresponding parts of congruent triangles are congruent, $\overline{AB} \cong \overline{A'B'}$. Therefore, $AB = A'B'$ by the definition of congruent segments.

A similar proof can be given when A and B are on opposite sides of the line k. The proof is left to the student. (See exercise 11.) ■

Since distance is preserved under a line reflection, we can prove that the image of a triangle is a congruent triangle and that angle measure, collinearity, and midpoint are also preserved. For each of the following corollaries, the image of A is A', the image of B is B', and the image of C is C'. Therefore, in the diagram, $\triangle ABC \cong \triangle A'B'C'$ by SSS. *We will use this fact to prove the following corollaries:*



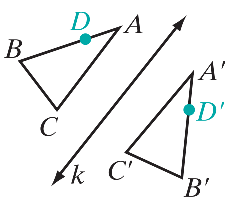
Corollary 6.1a

Under a line reflection, angle measure is preserved.

Proof: We found that $\angle ABC \cong \angle A'B'C'$. Therefore, $\angle ABC \cong \angle A'B'C'$ because the angles are corresponding parts of congruent triangles. ☐

Corollary 6.1b

Under a line reflection, collinearity is preserved.



Proof: Let D be a point on $\overline{AB}$ whose image is D'. Since distance is preserved:

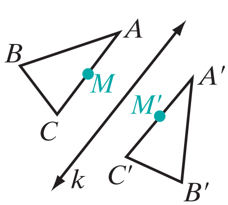
$$AD = A'D' \qquad DB = D'B' \qquad AB = A'B'$$

A, D, and B are collinear, D is between A and B, and $AD + DB = AB$. By substitution, $A'D' + D'B' = A'B'$.

If D' were not on $\overline{A'B'}$, $A'D' + D'B' > A'B'$ because a straight line is the shortest distance between two points. But by substitution, $A'D' + D'B' = A'B'$. Therefore, A', D' and B' are collinear and D' is between A' and B'. ☐

Corollary 6.1c

Under a line reflection, midpoint is preserved.



Proof: Let M be the midpoint of $\overline{AC}$ and M' the image of M. Since distance is preserved under a line reflection, $A'M' = AM$, and $M'C' = MC$.

Since M is the midpoint of $\overline{AC}$, $AM = MC$ and, by the substitution postulate, $A'M' = M'C'$.

Therefore, M' is the midpoint of $\overline{A'C'}$, that is, midpoint is preserved under a line reflection. ☐

We can summarize Theorem 6.1 and its corollaries in the following statement:

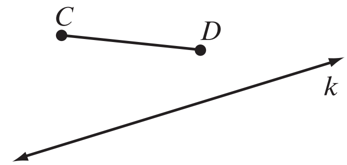
► **Under a line reflection, distance, angle measure, collinearity, and midpoint are preserved.**

We use r_k as a symbol for the image under a reflection in line k. For example,

$r_k(A) = B$	*means*	"The image of A under a reflection in line k is B."
$r_k(\triangle ABC) = \triangle A'B'C'$	*means*	"The image of $\triangle ABC$ under a reflection in line k is $\triangle A'B'C'$."

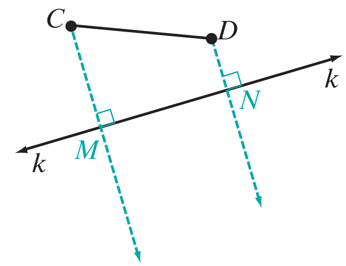
EXAMPLE 1

If $r_k(\overline{CD}) = \overline{C'D'}$, construct $\overline{C'D'}$.

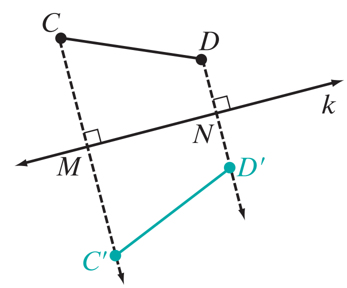


Construction

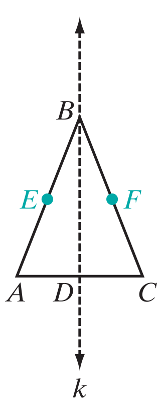
1. Construct the perpendicular line from C to k. Let the point of intersection be M.
2. Construct the perpendicular line from D to k. Let the point of intersection be N.



3. Construct point C' on $\overleftrightarrow{CM}$ such that $CM = MC'$ and point D' on $\overleftrightarrow{DN}$ such that $DN = ND'$.
4. Draw $\overline{C'D'}$.



Line Symmetry



We know that the altitude to the base of an isosceles triangle is also the median and the perpendicular bisector of the base. If we imagine that isosceles triangle ABC, shown at the left, is folded along the perpendicular bisector of the base so that A falls on C, the line along which it folds, k, is a reflection line. Every point of the triangle has as its image a point of the triangle. Points B and D are fixed points because they are points of the line of reflection.

Thus, under the line reflection in k:

1. All points of $\triangle ABC$ are reflected so that

$$A \to C \quad C \to A \quad E \to F \quad F \to E \quad B \to B \quad D \to D$$

2. The sides of $\triangle ABC$ are reflected; that is, $\overline{AB} \to \overline{CB}$, a statement verifying that the legs of an isosceles triangle are congruent. Also, $\overline{AC} \to \overline{CA}$, showing that the base is its own image.

3. The angles of $\triangle ABC$ are reflected; that is, $\angle BAD \rightarrow \angle BCD$, a statement verifying that the base angles of an isosceles triangle are congruent. Also, $\angle ABC \rightarrow \angle CBA$, showing that the vertex angle is its own image.

We can note some properties of a line reflection by considering the reflection of isosceles triangle ABC in line k:

1. Distance is preserved (unchanged).

$$\overline{AB} \rightarrow \overline{CB} \text{ and } AB = CB \qquad \overline{AD} \rightarrow \overline{CD} \text{ and } AD = CD$$

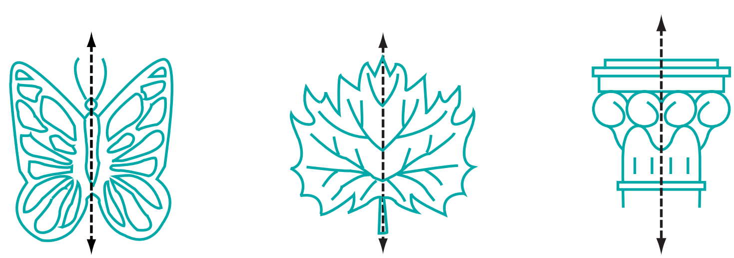
2. Angle measure is preserved.

$$\angle BAD \rightarrow \angle BCD \text{ and } m\angle BAD = m\angle BCD$$
$$\angle BDA \rightarrow \angle BDC \text{ and } m\angle BDA = m\angle BDC$$

3. The line of reflection is the perpendicular bisector of every segment joining a point to its image. The line of reflection, $\overleftrightarrow{BD}$, is the perpendicular bisector of $\overline{AC}$.

4. A figure is always congruent to its image: $\triangle ABC \cong \triangle CBA$.

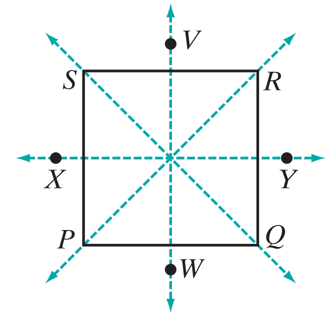
In nature, in art, and in industry, many forms have a pleasing, attractive appearance because of a balanced arrangement of their parts. We say that such forms have symmetry.



In each of the figures above, there is a line on which the figure could be folded so that the parts of the figure on opposite sides of the line would coincide. If we think of that line as a line of reflection, each point of the figure has as its image a point of the figure. This line of reflection is a line of symmetry, or an **axis of symmetry**, and the figure has *line symmetry*.

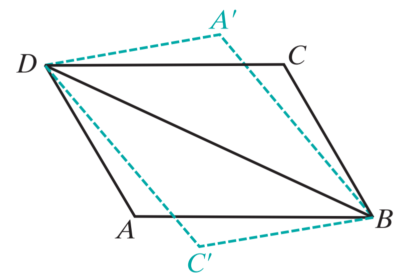
DEFINITION

A figure has **line symmetry** when the figure is its own image under a line reflection.



It is possible for a figure to have more than one axis of symmetry. In the square $PQRS$ at the right, $\overleftrightarrow{XY}$ is an axis of symmetry and $\overleftrightarrow{VW}$ is a second axis of symmetry. The diagonals, $\overline{PR}$ and $\overline{QS}$, are also segments of axes of symmetry.

Lines of symmetry may be found for some letters and for some words, as shown at the left.



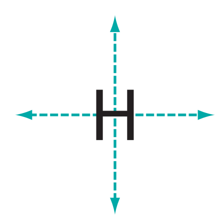
Not every figure, however, has line symmetry. If the parallelogram $ABCD$ at the right is reflected in the line that contains the diagonal $\overline{BD}$, the image of A is A' and the image of C is C'. Points A' and C', however, are not points of the original parallelogram. The image of parallelogram $ABCD$ under a reflection in $\overleftrightarrow{BD}$ is $A'BC'D$. Therefore, $ABCD$ is not symmetric with respect to $\overleftrightarrow{BD}$.

We have used the line that contains the diagonal $\overline{BD}$ as a line of reflection, but note that it is not a line of symmetry. There is no line along which the parallelogram can be folded so that the image of every point of the parallelogram will be a point of the parallelogram under a reflection in that line.

EXAMPLE 2

How many lines of symmetry does the letter H have?

Solution The horizontal line through the crossbar is a line of symmetry. The vertical line midway between the vertical segments is also a line of symmetry.



Answer The letter H has two lines of symmetry.

Exercises

Writing About Mathematics

1. Explain how a number line illustrates a one-to-one correspondence between the set of points on a line and the set of real numbers.

2. Is the correspondence $(x, y) \rightarrow (2, y)$ a transformation? Explain why or why not.

Developing Skills

3. If $r_k(\triangle PQR) = \triangle P'Q'R'$ and $\triangle PQR$ is isosceles with $PQ = QR$, prove that $\triangle P'Q'R'$ is isosceles.

4. If $r_k(\triangle LMN) = \triangle L'M'N'$ and $\triangle LMN$ is a right triangle with $m\angle N = 90$, prove that $\triangle L'M'N'$ is a right triangle.

In 5–9, under a reflection in line k, the image of A is A', the image of B is B', the image of C is C', and the image of D is D.

5. Is $\triangle ABC \cong \triangle A'B'C'$? Justify your answer.

6. If $\overline{AC} \perp \overline{BC}$, is $\overline{A'C'} \perp \overline{B'C'}$? Justify your answer?

7. The midpoint of $\overline{AB}$ is M. If the image of M is M', is M' the midpoint of $\overline{A'B'}$? Justify your answer.

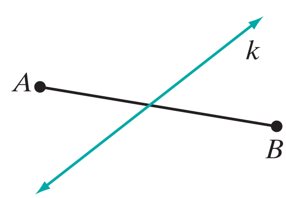
8. If A, M, and B lie on a line, do A', M', and B' lie on a line? Justify your answer.

9. Which point lies on the line of reflection? Justify your answer.

10. A triangle, $\triangle RST$, has line symmetry with respect to the altitude from S but does not have line symmetry with respect to the altitudes from R and T. What kind of a triangle is $\triangle RST$? Justify your answer.

Applying Skills

11. Write the proof of Theorem 6.1 for the case when points A and B are on opposite sides of line k.



12. A baseball diamond is in the shape of a rhombus with sides 90 feet in length. Describe the lines of symmetry of a baseball diamond in terms of home plate and the three bases.

13. Print three letters that have line symmetry with respect to only one line and draw the line of symmetry.

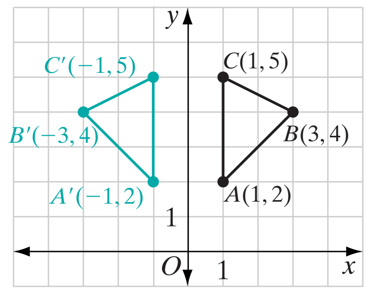
14. Print three letters that have line symmetry with respect to exactly two lines and draw the lines of symmetry.

6-3 LINE REFLECTIONS IN THE COORDINATE PLANE

We can apply the definition of a line reflection to points in the coordinate plane.

Reflection in the *y*-axis

In the figure, $\triangle ABC$ is reflected in the y-axis. Its image under the reflection is $\triangle A'B'C'$. From the figure, we see that:



$$A(1, 2) \rightarrow A'(-1, 2)$$
$$B(3, 4) \rightarrow B'(-3, 4)$$
$$C(1, 5) \rightarrow C'(-1, 5)$$

For each point and its image under a reflection in the y-axis, the y-coordinate of the image is the same as the y-coordinate of the point; the x-coordinate of the image is the opposite of the x-coordinate of the point. Note that for a reflection in the y-axis, the image of $(1, 2)$ is $(-1, 2)$ and the image of $(-1, 2)$ is $(1, 2)$.

A reflection in the y-axis can be designated as $r_{y\text{-axis}}$. For example, if the image of $(1, 2)$ is $(-1, 2)$ under a reflection in the y-axis, we can write:

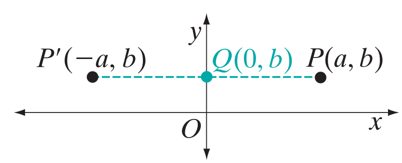
$$r_{y\text{-axis}}(1, 2) = (-1, 2)$$

From these examples, we form a general rule that can be proven as a theorem.

Theorem 6.2 Under a reflection in the y-axis, the image of $P(a, b)$ is $P'(-a, b)$.

Given A reflection in the y-axis.

Prove The image of $P(a, b)$ under a reflection in the y-axis is $P'(-a, b)$.



Proof By the definition of a reflection in a line, a point P' is the image of P under a reflection in a given line if and only if the line is the perpendicular bisector of $\overline{PP'}$. Therefore, we can prove that P' is the image of P under a reflection in the y-axis by showing that the y-axis is the perpendicular bisector of $\overline{PP'}$.

(1) *The y-axis is perpendicular to* $\overline{PP'}$. The line of reflection, the y-axis, is a vertical line. $P(a, b)$ and $P'(-a, b)$ have the same y-coordinates. $\overline{PP'}$ is a segment of a horizontal line because two points are on the same horizontal line if and only if they have the same y-coordinates. Every vertical line is perpendicular to every horizontal line. Therefore, the y-axis is perpendicular to $\overline{PP'}$.

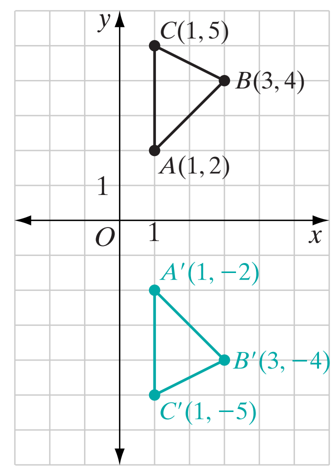
(2) *The y-axis bisects* $\overline{PP'}$. Let Q be the point at which $\overline{PP'}$ intersects the y-axis. The x-coordinate of every point on the y-axis is 0. The length of a horizontal line segment is the absolute value of the difference of the x-coordinates of the endpoints.

$$PQ = |a - 0| = |a| \qquad \text{and} \qquad P'Q = |-a - 0| = |a|$$

Since $PQ = P'Q$, Q is the midpoint of $\overline{PP'}$ or the y-axis bisects $\overline{PP'}$.

Steps 1 and 2 prove that if P has the coordinates (a, b) and P' has the coordinates $(-a, b)$, the y-axis is the perpendicular bisector of $\overline{PP'}$, and therefore, the image of $P(a, b)$ is $P'(-a, b)$. ❑

Reflection in the *x*-axis



In the figure, $\triangle ABC$ is reflected in the x-axis. Its image under the reflection is $\triangle A'B'C'$. From the figure, we see that:

$$A(1, 2) \rightarrow A'(1, -2)$$
$$B(3, 4) \rightarrow B'(3, -4)$$
$$C(1, 5) \rightarrow C'(1, -5)$$

For each point and its image under a reflection in the x-axis, the x-coordinate of the image is the same as the x-coordinate of the point; the y-coordinate of the image is the opposite of the y-coordinate of the point. Note that for a reflection in the x-axis, the image of (1, 2) is (1, −2) and the image of (1, −2) is (1, 2).

A reflection in the x-axis can be designated as $r_{x\text{-axis}}$. For example, if the image of (1, 2) is (1, −2) under a reflection in the x-axis, we can write:

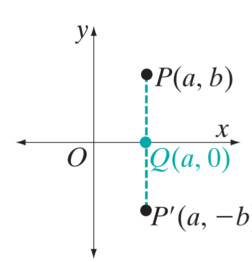
$$r_{x\text{-axis}}(1, 2) = (1, -2)$$

From these examples, we form a general rule that can be proven as a theorem.

Theorem 6.3

> Under a reflection in the x-axis, the image of $P(a, b)$ is $P'(a, -b)$.

The proof follows the same general pattern as that for a reflection in the y-axis. Prove that the x-axis is the perpendicular bisector of $\overline{PP'}$. The proof is left to the student. (See exercise 18.)



EXAMPLE 1

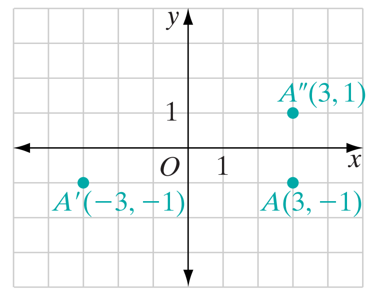
On graph paper:

a. Locate $A(3, -1)$.

b. Locate A', the image of A under a reflection in the y-axis, and write its coordinates.

c. Locate A'', the image of A under a reflection in the x-axis, and write its coordinates.

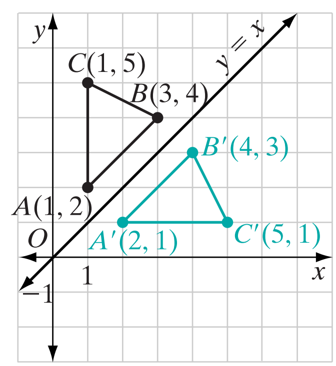
Answers



Reflection in the Line $y = x$

In the figure, $\triangle ABC$ is reflected in the line $y = x$. Its image under the reflection is $\triangle A'B'C'$. From the figure, we see that:

$$A(1, 2) \rightarrow A'(2, 1)$$
$$B(3, 4) \rightarrow B'(4, 3)$$
$$C(1, 5) \rightarrow C'(5, 1)$$



For each point and its image under a reflection in the line $y = x$, the x-coordinate of the image is the y-coordinate of the point; the y-coordinate of the image is the x-coordinate of the point. Note that for a reflection in the line $y = x$, the image of $(1, 2)$ is $(2, 1)$ and the image of $(2, 1)$ is $(1, 2)$.

A reflection in the line $y = x$ can be designated as $r_{y=x}$. For example, if the image of $(1, 2)$ is $(2, 1)$ under a reflection in $y = x$, we can write:

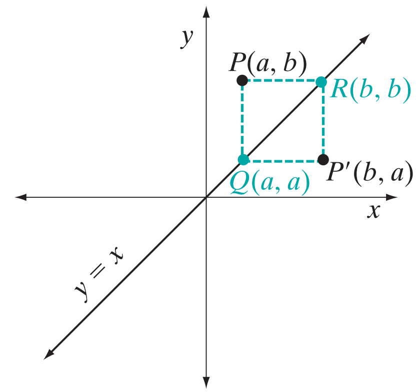
$$r_{y=x}(1, 2) = (2, 1)$$

From these examples, we form a general rule that can be proven as a theorem.

Theorem 6.4 Under a reflection in the line $y = x$, the image of $P(a, b)$ is $P'(b, a)$.

Given A reflection in the line whose equation is $y = x$.

Prove The image of $P(a, b)$ is $P'(b, a)$.



Proof For every point on the line $y = x$, the x-coordinate and the y-coordinate are equal. Locate the points $Q(a, a)$ and $R(b, b)$ on the line $y = x$.

If two points have the same x-coordinate, then the distance between them is the absolute value of their y-coordinates and if two points have the same y-coordinate, then the distance between them is the absolute value of their x-coordinates.

$$PQ = |b - a| \text{ and } P'Q = |b - a|$$

Therefore, Q is equidistant from P and P'.

$$PR = |b - a| \text{ and } P'R = |b - a|$$

Therefore, R is equidistant from P and P'.

If two points are each equidistant from the endpoints of a line segment, they lie on the perpendicular bisector of the line segment (Theorem 5.3). Therefore, $\overline{RQ}$, which is a subset of the line $y = x$, is the perpendicular bisector of $\overline{PP'}$. By the definition of a line reflection across the line $y = x$, the image of $P(a, b)$ is $P'(b, a)$.

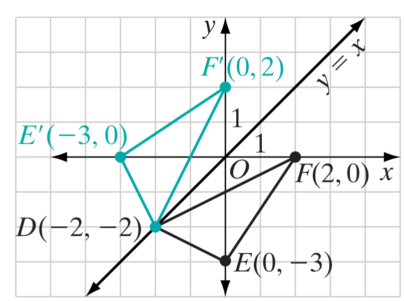
EXAMPLE 2

The vertices of $\triangle DEF$ are $D(-2, -2)$, $E(0, -3)$, and $F(2, 0)$.

a. Draw $\triangle DEF$ on graph paper.

b. Draw the image of $\triangle DEF$ under a reflection in the line whose equation is $y = x$.

Answers



Exercises

Writing About Mathematics

1. When finding the distance from $P(a, b)$ to $Q(a, a)$, Allison wrote $PQ = |b - a|$ and Jacob wrote $PQ = |a - b|$. Explain why both are correct.

2. The image of point A is in the third quadrant under a reflection in the y-axis and in the first quadrant under a reflection in the x-axis. In what quadrant is the image of A under a reflection in the line $y = x$?

Developing Skills

In 3–7: **a.** On graph paper, locate each point and its image under $r_{x\text{-axis}}$. **b.** Write the coordinates of the image point.

3. $(2, 5)$ **4.** $(1, 3)$ **5.** $(-2, 3)$ **6.** $(2, -4)$ **7.** $(0, 2)$

In 8–12: **a.** On graph paper, locate each point and its image under $r_{y\text{-axis}}$. **b.** Write the coordinates of the image point.

8. $(3, 5)$ **9.** $(1, 4)$ **10.** $(2, -3)$ **11.** $(-2, 3)$ **12.** $(-1, 0)$

In 13–17: **a.** On graph paper, locate each point and its image under $r_{y=x}$. **b.** Write the coordinates of the image point.

13. $(3, 5)$ **14.** $(-3, 5)$ **15.** $(4, -2)$ **16.** $(-1, -5)$ **17.** $(2, 2)$

Applying Skills

18. Prove Theorem 6.3, "Under a reflection in the x-axis, the image of $P(a, b)$ is $P'(a, -b)$."

19. When the points $A(-4, 0)$, $B(0, -4)$, $C(4, 0)$ and $D(0, 4)$ are connected in order, square $ABCD$ is drawn.

a. Show that the line $y = x$ is a line of symmetry for the square.

b. Show that the y-axis is a line of symmetry for the square.

20. Show that the y-axis is not a line of symmetry for the rectangle whose vertices are $E(0, -3)$, $F(5, -3)$, $G(5, 3)$, and $H(0, 3)$.

21. Write the equation of two lines that are lines of symmetry for the rectangle whose vertices are $E(0, -3)$, $F(6, -3)$, $G(6, 3)$, and $H(0, 3)$.

Hands-On Activity 1

In this activity, you will learn how to construct a reflection in a line using a compass and a straightedge, or geometry software. (*Note:* Compass and straightedge constructions can also be done on the computer by using only the point, line segment, line, and circle creation tools of your geometry software and no other software tools.)

STEP 1. Draw a line segment. Label the endpoints A and B. Draw a reflection line k.

STEP 2. Construct line l perpendicular to line k through point A. Let M be the point where lines l and k intersect.

STEP 3. Construct line segment $\overline{A'M}$ congruent to $\overline{AM}$ along line l. Using point M as the center, draw a circle with radius equal to AM. Let A' be the point where the circle intersects line l on the ray that is the opposite ray of $\overrightarrow{MA}$.

STEP 4. Repeat steps 2 and 3 for point B in order to construct B'.

STEP 5. Draw $\overline{A'B'}$.

Result: $\overline{A'B'}$ is the image of $\overline{AB}$ under a reflection in line k.

For each figure, construct the reflection in the given line.

a. Segment $\overline{AB}$ with vertices $A(-4, 2)$ and $B(2, 4)$, and line k through points $C(-2, 1)$ and $D(0, 5)$.

b. Angle EFG with vertices $E(4, 3)$, $F(1, 6)$, and $G(6, 6)$, and line l through points $H(1, 3)$ and $I(3, 1)$.

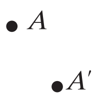
c. Triangle JKL with vertices $J(-2, 2)$, $K(-5, 1)$, and $L(-1, 4)$, and line p through points $M(-7, -3)$ and $N(2, 0)$.

d. Segments $\overline{PQ}$ and $\overline{RS}$ with vertices $P(-3, 4)$, $Q(3, 2)$, $R(-1, 0)$, and $S(1, 6)$, and line q through points $T(0, -3)$ and $U(3, 0)$.

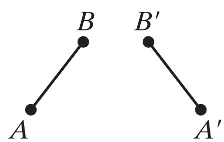
Hands-On Activity 2

In each figure, one is the image of the other under a reflection in a line. For each figure, construct the reflection line using a compass and a straightedge.

a.



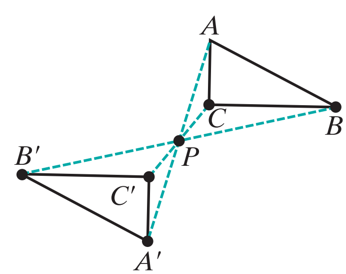
b.



c. Triangle ABC with vertices $A(-1, 4)$, $B(4, 3)$, and $(-2, 8)$, and triangle $A'B'C'$ with vertices $A'(-3, 2)$, $B'(-2, -3)$, $C'(-7, 3)$.

d. Triangle ABC with vertices $A(0, 4)$, $B(7, 1)$, and $C(6, 6)$, and triangle $A'B'C'$ with vertices $A'(7, -3)$, $B'(4, 4)$, $C'(9, 3)$.

6-4 POINT REFLECTIONS IN THE COORDINATE PLANE



The figure at the right illustrates another type of reflection, a reflection in a point. In the figure, $\triangle A'B'C'$ is the image of $\triangle ABC$ under a reflection in point P. If a line segment is drawn connecting any point to its image, then the point of reflection is the midpoint of that segment. In the figure:

- Point A' is on $\overleftrightarrow{AP}$, $AP = PA'$, and P is the midpoint of $\overline{AA'}$.
- Point B' is on $\overleftrightarrow{BP}$, $BP = PB'$, and P is the midpoint of $\overline{BB'}$.
- Point C' is on $\overleftrightarrow{CP}$, $CP = PC'$, and P is the midpoint of $\overline{CC'}$.

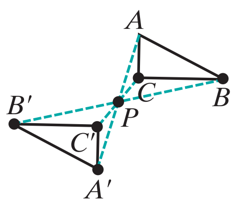
DEFINITION

A **point reflection in *P*** is a transformation of the plane such that:

1. If point A is not point P, then the image of A is A' and P the midpoint of $\overline{AA'}$.

2. The point P is its own image.

Properties of Point Reflections

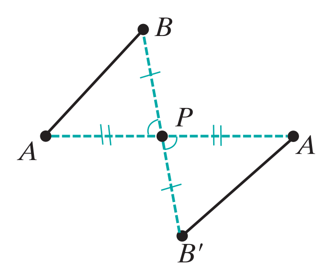


Looking at triangles ABC and $A'B'C'$ and point of reflection P, we observe some properties of a point reflection:

1. Distance is preserved: $AB = A'B'$.
2. Angle measure is preserved: $m\angle ACB = m\angle A'C'B'$.
3. A figure is always congruent to its image.
4. Collinearity is preserved. The image of any point on $\overline{AB}$ is a point on $\overline{A'B'}$.
5. Midpoint is preserved. The image of the midpoint of $\overline{AB}$ is the midpoint of $\overline{A'B'}$.

We can state the first property as a theorem and the remaining properties as corollaries to this theorem.

Theorem 6.5 Under a point reflection, distance is preserved.



Given Under a reflection in point P, the image of A is A' and the image of B is B'.

Prove $AB = A'B'$

Proof Since $\angle APB$ and $\angle A'PB'$ are vertical angles, $\angle APB \cong \angle A'PB'$. In a point reflection, if a point X is not point P, then the image of X is X' and P the midpoint of $\overline{XX'}$. Therefore, P is the midpoint of both $\overline{AA'}$ and $\overline{BB'}$, and $\overline{BP} \cong \overline{BP'}$ and $\overline{AP} \cong \overline{PA'}$. By SAS, $\triangle APB \cong \triangle A'PB'$ and $AB = A'B'$.

The case when either point A or B is point P is left to the student. (See exercise 9.)

Corollary 6.5a Under a point reflection, angle measure is preserved.

Corollary 6.5b Under a point reflection, collinearity is preserved.

Corollary 6.5c Under a point reflection, midpoint is preserved.

We can prove that angle measure, collinearity, and midpoint are preserved using the same proofs that we used to prove the corollaries of Theorem 6.1. (See exercises 10–12.)

Theorem 6.5 and its corollaries can be summarized in the following statement.

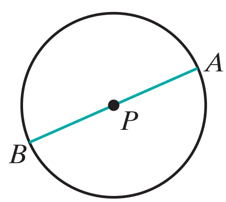
► **Under a point reflection, distance, angle measure, collinearity, and midpoint are preserved.**

We use R_P as a symbol for the image under a reflection in point P. For example,

$R_P(A) = B$ *means* "The image of A under a reflection in point P is B."

$R_{(1,2)}(A) = A'$ *means* "The image of A under a reflection in point (1, 2) is A'."

Point Symmetry



DEFINITION

A figure has **point symmetry** if the figure is its own image under a reflection in a point.

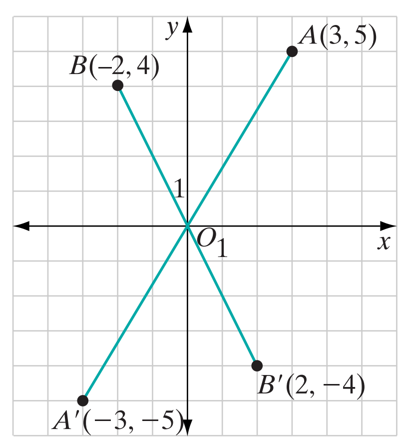
A circle is the most common example of a figure with point symmetry. Let P be the center of a circle, A be any point on the circle, and B be the other point at which $\overleftrightarrow{AP}$ intersect the circle. Since every point on a circle is equidistant from the center, $PA = PB$, P is the midpoint of $\overline{AB}$ and B, a point on the circle, is the image of A under a reflection in P.

Other examples of figures that have point symmetry are letters such as **S** and **N** and numbers such as **8**.

Point Reflection in the Coordinate Plane

In the coordinate plane, the origin is the most common point that is used to used to define a point reflection.

In the diagram, points $A(3, 5)$ and $B(-2, 4)$ are reflected in the origin. The coordinates of A', the image of A, are $(-3, -5)$ and the coordinates of B', the image of B, are $(2, -4)$. These examples suggest the following theorem.



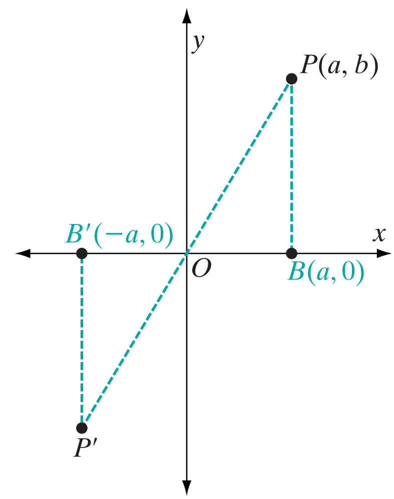
Theorem 6.6 Under a reflection in the origin, the image of $P(a, b)$ is $P'(-a, -b)$.

Given A reflection in the origin.

Prove Under a reflection in the origin, the image of $P(a, b)$ is $P'(-a, -b)$.

Proof Let P' be the image of $P(a, b)$ under a reflection in the origin, O. Then:

$$OP = OP' \text{ and } \overline{OP} \cong \overline{OP'}$$



Let $B(a, 0)$ be the point of intersection of the x-axis and a vertical line through P. Then:

$$OB = |0 - a| = |a|$$

Let B' be the point $(-a, 0)$. Then:

$$OB' = |0 - (-a)| = |a|$$
$$OB = OB' \text{ and } \overline{OB} \cong \overline{OB'}$$

$\angle POB \cong \angle P'OB'$ because vertical angles are congruent. Therefore, $\triangle POB \cong \triangle P'OB'$ by SAS. In particular,

$$\begin{aligned} PB = P'B' &= |0 - b| \\ &= |b| \end{aligned}$$

Since $\angle OBP$ is a right angle, $\angle OB'P'$ is a right angle and $\overline{P'B'}$ is a vertical line. Therefore, P' has the same x-coordinate as B' and is b units in the opposite direction from the x-axis as P. The coordinates of P' are $(-a, -b)$. ∎

Note: $-a$ and $-b$ are the opposites of a and b. The image of $(-4, 2)$ is $(4, -2)$ and the image of $(-5, -1)$ is $(5, 1)$.

We can symbolize the reflection in the origin as R_O. Therefore, we can write:

$$R_O(a, b) = (-a, -b)$$

EXAMPLE 1

a. What are the coordinates of B, the image of $A(-3, 2)$ under a reflection in the origin?

b. What are the coordinates of C, the image of $A(-3, 2)$ under a reflection in the x-axis?

c. What are the coordinates of D, the image of C under a reflection in the y-axis?

d. Does a reflection in the origin give the same result as a reflection in the x-axis followed by a reflection in the y-axis? Justify your answer.

Solution **a.** Since $R_O(a, b) = (-a, -b)$, $R_O(-3, 2) = (3, -2)$. The coordinates of B are $(3, -2)$. *Answer*

b. Since $r_{x\text{-axis}}(a, b) = (a, -b)$, $r_{x\text{-axis}}(-3, 2) = (-3, -2)$. The coordinates of C are $(-3, -2)$. *Answer*

c. Since $r_{y\text{-axis}}(a, b) = (-a, b)$, $r_{y\text{-axis}}(-3, -2) = (3, -2)$. The coordinates of D are $(3, -2)$. *Answer*

d. For the point $A(-3, 2)$, a reflection in the origin gives the same result as a reflection in the x-axis followed by a reflection in the y-axis. In general,

$$R_O(a, b) = (-a, -b)$$

while

$$r_{x\text{-axis}}(a, b) = (a, -b)$$
$$r_{y\text{-axis}}(a, -b) = (-a, -b)$$

Therefore, a reflection in the origin gives the same result as a reflection in the x-axis followed by a reflection in the y-axis. *Answer*

Exercises

Writing About Mathematics

1. Ada said if the image of $\overline{AB}$ under a reflection in point P is $\overline{A'B'}$, then the image of $\overline{A'B'}$ under a reflection in point P is $\overline{AB}$. Do you agree with Ada? Justify your answer.

2. Lines l and m intersect at P. $\overline{A'B'}$ is the image of $\overline{AB}$ under a reflection in line l and $\overline{A''B''}$ is the image of $\overline{A'B'}$ under a reflection in line m. Is the result the same if $\overline{AB}$ is reflected in P? Justify your answer.

Developing Skills

In 3–8, give the coordinates of the image of each point under R_O.

3. $(1, 5)$ **4.** $(-2, 4)$ **5.** $(-1, 0)$ **6.** $(0, 3)$ **7.** $(6, 6)$ **8.** $(-1, -5)$

Applying Skills

9. Write a proof of Theorem 6.5 for the case when either point A or B is point P, that is, when the reflection is in one of the points.

10. Prove Corollary 6.5a, "Under a point reflection, angle measure is preserved."

11. Prove Corollary 6.5b, "Under a point reflection, collinearity is preserved."

12. Prove Corollary 6.5c, "Under a point reflection, midpoint is preserved."

13. The letters **S** and **N** have point symmetry. Print 5 other letters that have point symmetry.

14. Show that the quadrilateral with vertices $P(-5, 0)$, $Q(0, -5)$, $R(5, 0)$, and $S(0, 5)$ has point symmetry with respect to the origin.

15. **a.** What is the image of $A(2, 6)$ under a reflection in $P(2, 0)$?

b. What is the image of $B(3, 6)$ under a reflection in $P(2, 0)$?

c. Is the reflection in the point $P(2, 0)$ the same as the reflection in the x-axis? Justify your answer.

16. **a.** What is the image of $A(4, 4)$ under a reflection in the point $P(4, -2)$?

b. What is the image of $C(1, -2)$ under a reflection in the point $P(4, -2)$?

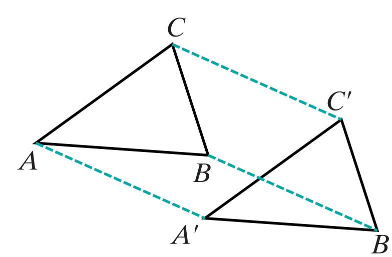
c. The point $D(2, 0)$ lies on the segment $\overline{AB}$. Does the image of D lie on the image of $\overline{AB}$? Justify your answer.

6-5 TRANSLATIONS IN THE COORDINATE PLANE

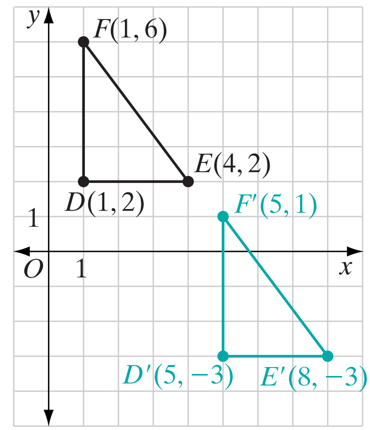
It is often useful or necessary to move objects from one place to another. If we move a table from one place in the room to another, the bottom of each leg moves the same distance in the same direction.

DEFINITION

A **translation** is a transformation of the plane that moves every point in the plane the same distance in the same direction.



If $\triangle A'B'C'$ is the image of $\triangle ABC$ under a translation, $AA' = BB' = CC'$. It appears that the size and shape of the figure are unchanged, so that $\triangle ABC \cong \triangle A'B'C'$. Thus, under a translation, as with a reflection, a figure is congruent to its image. The following example in the coordinate plane shows that this is true. In the coordinate plane, the distance is given in terms of horizontal distance (change in the x-coordinates) and vertical distance (change in the y-coordinates).



In the figure, $\triangle DEF$ is translated by moving every point 4 units to the right and 5 units down. From the figure, we see that:

$$D(1, 2) \rightarrow D'(5, -3)$$

$$E(4, 2) \rightarrow E'(8, -3)$$

$$F(1, 6) \rightarrow F'(5, 1)$$

- $\overline{DE}$ and $\overline{D'E'}$ are horizontal segments.

$$DE = |1 - 4| = 3 \qquad D'E' = |5 - 8| = 3$$

- $\overline{DF}$ and $\overline{D'F'}$ are vertical segments.

$$DF = |2 - 6| = 4 \qquad D'F' = |-3 - 1| = 4$$

- $\angle FDE$ and $\angle F'D'E'$ are right angles.

Therefore, $\triangle DEF \cong \triangle D'E'F'$ by SAS.

This translation moves every point 4 units to the right (+4) and 5 units down (−5).

1. The x-coordinate of the image is 4 more than the x-coordinate of the point:

$$x \rightarrow x + 4$$

2. The y-coordinate of the image is 5 less than the y-coordinate of the point:

$$y \rightarrow y - 5$$

From this example, we form a general rule:

DEFINITION

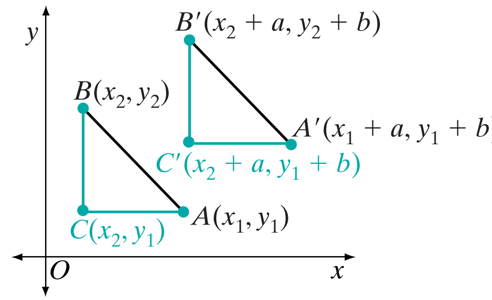
A **translation of *a* units in the horizontal direction and *b* units in the vertical direction** is a transformation of the plane such that the image of $P(x, y)$ is

$$P'(x + a, y + b).$$

Note: If the translation moves a point to the right, a is positive; if it moves a point to the left, a is negative; if the translation moves a point up, b is positive; if it moves a point down, b is negative.

Theorem 6.7 Under a translation, distance is preserved.

Given A translation in which the image of $A(x_1, y_1)$ is $A'(x_1 + a, y_1 + b)$ and the image of $B(x_2, y_2)$ is $B'(x_2 + a, y_2 + b)$.



Prove $AB = A'B'$

Proof Locate $C(x_2, y_1)$, a point on the same horizontal line as A and the same vertical line as B. Then by definition, the image of C is

$$C'(x_2 + a, y_1 + b).$$

Thus:

$$\begin{aligned} BC &= |y_2 - y_1| \\ B'C' &= |(y_2 + b) - (y_1 + b)| \\ &= |y_2 - y_1 + b - b| \\ &= |y_2 - y_1| \end{aligned} \qquad \begin{aligned} AC &= |x_1 - x_2| \\ A'C' &= |(x_1 + a) - (x_2 + a)| \\ &= |x_1 - x_2 + a - a| \\ &= |x_1 - x_2| \end{aligned}$$

Since horizontal and vertical lines are perpendicular, $\angle BCA$ and $\angle B'C'A'$ are both right angles. By SAS, $\triangle ABC \cong \triangle A'B'C'$. Then, AB and $A'B'$ are the equal measures of the corresponding sides of congruent triangles. ∎

When we have proved that distance is preserved, we can prove that angle measure, collinearity, and midpoint are preserved. The proofs are similar to the proofs of the corollaries of Theorem 6.1 and are left to the student. (See exercise 10.)

Corollary 6.7a Under a translation, angle measure is preserved.

Corollary 6.7b Under a translation, collinearity is preserved.

Corollary 6.7c Under a translation, midpoint is preserved.

We can write Theorem 6.7 and its corollaries as a single statement.

▶ **Under a translation, distance, angle measure, collinearity, and midpoint are preserved.**

Let $T_{a,b}$ be the symbol for a translation of a units in the horizontal direction and b units in the vertical direction. We can write:

$$T_{a,b}(x, y) = (x + a, y + b)$$

EXAMPLE 1

The coordinates of the vertices of $\triangle ABC$ are $A(3, -3)$, $B(3, 2)$, and $C(6, 1)$.

a. Find the coordinates of the vertices of $\triangle A'B'C'$, the image of $\triangle ABC$ under $T_{-5,3}$.

b. Sketch $\triangle ABC$ and $\triangle A'B'C'$ on graph paper.

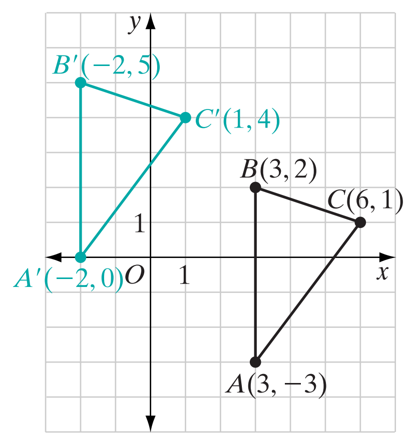
Solution **a.** Under the given translation, every point moves 5 units to the left and 3 units up.

$A(3, -3) \rightarrow A'(3 - 5, -3 + 3) = A'(-2, 0)$

$B(3, 2) \rightarrow B'(3 - 5, 2 + 3) = B'(-2, 5)$

$C(6, 1) \rightarrow C'(6 - 5, 1 + 3) = C'(1, 4)$

b.



Translational Symmetry

DEFINITION

A figure has **translational symmetry** if the image of every point of the figure is a point of the figure.

Patterns used for decorative purposes such as wallpaper or borders on clothing often appear to have *translational symmetry*. True translational symmetry would be possible, however, only if the pattern could repeat without end.

Exercises

Writing About Mathematics

1. Explain why there can be no fixed points under a translation other than $T_{0,0}$.

2. Hunter said that if A' is the image of A under a reflection in the y-axis and A'' is the image of A' under a reflection in the line $x = 3$, then A'' is the image of A under the translation $(x, y) \to (x + 6, y)$.

a. Do you agree with Hunter when A is a point with an x-coordinate greater than or equal to 3? Justify your answer.

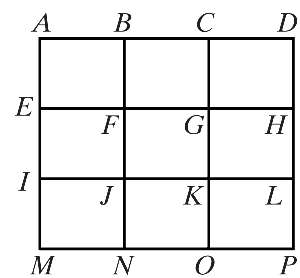
b. Do you agree with Hunter when A is a point with an x-coordinate greater than 0 but less than 3? Justify your answer.

c. Do you agree with Hunter when A is a point with a negative x-coordinate? Justify your answer.

Developing Skills

3. The diagram consists of nine congruent rectangles. Under a translation, the image of A is G. Find the image of each of the given points under the same translation.

a. J **b.** B **c.** I **d.** F **e.** E



4. a. On graph paper, draw and label $\triangle ABC$, whose vertices have the coordinates $A(1, 2)$, $B(6, 3)$, and $C(4, 6)$.

b. Under the translation $P(x, y) \to P'(x + 5, y - 3)$, the image of $\triangle ABC$ is $\triangle A'B'C'$. Find the coordinates of A', B', and C'.

c. On the same graph drawn in part **a**, draw and label $\triangle A'B'C'$.

5. a. On graph paper, draw and label $\triangle ABC$ if the coordinates of A are $(-2, -2)$, the coordinates of B are $(2, 0)$, and the coordinates of C are $(3, -3)$.

b. On the same graph, draw and label $\triangle A'B'C'$, the image of $\triangle ABC$ under the translation $T_{-4,7}$.

c. Give the coordinates of the vertices of $\triangle A'B'C'$.

6. If the rule of a translation is written $(x, y) \to (x + a, y + b)$, what are the values of a and b for a translation where every point moves 6 units to the right on a graph?

7. In a translation, every point moves 4 units down. Write a rule for this translation in the form $(x, y) \to (x + a, y + b)$.

8. The coordinates of $\triangle ABC$ are $A(2, 1)$, $B(4, 1)$, and $C(5, 5)$.

a. On graph paper, draw and label $\triangle ABC$.

b. Write a rule for the translation in which the image of A is $C(5, 5)$.

c. Use the rule from part **b** to find the coordinates of B', the image of B, and C', the image of C, under this translation.

d. On the graph drawn in part **a**, draw and label $\triangle C'B'C'$, the image of $\triangle ABC$.

9. The coordinates of the vertices of $\triangle ABC$ are $A(0, -1)$, $B(2, -1)$, and $C(3, 3)$.

a. On graph paper, draw and label $\triangle ABC$.

b. Under a translation, the image of C is $B(2, -1)$. Find the coordinates of A', the image of A, and of B', the image of B, under this same translation.

c. On the graph drawn in part **a**, draw and label $\triangle A'B'B'$, the image of $\triangle ABC$.

d. How many points, if any, are fixed points under this translation?

Applying Skills

10. Prove the corollaries of Theorem 6.7.

a. Corollary 6.7a, "Under a translation, angle measure is preserved."

b. Corollary 6.7b, "Under a translation, collinearity is preserved."

c. Corollary 6.7c, "Under a translation, midpoint is preserved."

(*Hint:* See the proofs of the corollaries of Theorem 6.1 on page 217.)

11. The coordinates of the vertices of $\triangle LMN$ are $L(-6, 0)$, $M(-2, 0)$, and $N(-2, 2)$.

a. Draw $\triangle LMN$ on graph paper.

b. Find the coordinates of the vertices of $\triangle L'M'N'$, the image of $\triangle LMN$ under a reflection in the line $x = -1$, and draw $\triangle L'M'N'$ on the graph drawn in **a**.

c. Find the coordinates of the vertices of $\triangle L''M''N''$, the image of $\triangle L'M'N'$ under a reflection in the line $x = 4$, and draw $\triangle L''M''N''$ on the graph drawn in **a**.

d. Find the coordinates of the vertices of $\triangle PQR$, the image of $\triangle LMN$ under the translation $T_{10,0}$.

e. What is the relationship between $\triangle L''M''N''$ and $\triangle PQR$?

12. The coordinates of the vertices of $\triangle DEF$ are $D(-2, -3)$, $E(1, -3)$, and $F(0, 0)$.

a. Draw $\triangle DEF$ on graph paper.

b. Find the coordinates of the vertices of $\triangle D'E'F'$, the image of $\triangle DEF$ under a reflection in the line $y = 0$ (the x-axis), and draw $\triangle D'E'F'$ on the graph drawn in **a**.

c. Find the coordinates of the vertices of $\triangle D''E''F''$, the image of $\triangle D'E'F'$ under a reflection in the line $y = 3$, and draw $\triangle D''E''F''$ on the graph drawn in **a**.

d. Find the coordinates of the vertices of $\triangle RST$, the image of $\triangle DEF$ under the translation $T_{0,6}$.

e. What is the relationship between $\triangle D''E''F''$ and $\triangle RST$?

13. The coordinates of the vertices of $\triangle ABC$ are $A(1, 2)$, $B(5, 2)$, and $C(4, 5)$.

a. Draw $\triangle ABC$ on graph paper.

b. Find the coordinates of the vertices of $\triangle A'B'C'$, the image of $\triangle ABC$ under a reflection in the line $y = 0$ (the x-axis), and draw $\triangle A'B'C'$ on the graph drawn in **a**.

c. Find the coordinates of the vertices of $\triangle A''B''C''$, the image of $\triangle A'B'C'$ under a reflection in the line $y = -3$, and draw $\triangle A''B''C''$ on the graph drawn in **a**.

d. Is there a translation, $T_{a,b}$, such that $\triangle A''B''C''$ is the image of $\triangle ABC$? If so, what are the values of a and b?

14. In exercises, 11, 12, and 13, is there a relationship between the distance between the two lines of reflection and the x-values and y-values in the translation?

15. The coordinates of the vertices of $\triangle ABC$ are $A(1, 2)$, $B(5, 2)$, and $C(4, 5)$.

a. Draw $\triangle ABC$ on graph paper.

b. Find the coordinates of the vertices of $\triangle A'B'C'$, the image of $\triangle ABC$ under a reflection in any horizontal line, and draw $\triangle A'B'C'$ on the graph drawn in **a**.

c. Find the coordinates of the vertices of $\triangle A''B''C''$, the image of $\triangle A'B'C'$ under a reflection in the line that is 3 units below the line of reflection that you used in **b**, and draw $\triangle A''B''C''$ on the graph drawn in **a**.

d. What is the translation $T_{a,b}$ such that $\triangle A''B''C''$ is the image of $\triangle ABC$? Is this the same translation that you found in exercise 13?

6-6 ROTATIONS IN THE COORDINATE PLANE

Think of what happens to all of the points of a wheel as the wheel is turned. Except for the fixed point in the center, every point moves through a part of a circle, or *arc*, so that the position of each point is changed by a *rotation* of the same number of degrees.

DEFINITION

A **rotation** is a transformation of a plane about a fixed point P through an angle of d degrees such that:

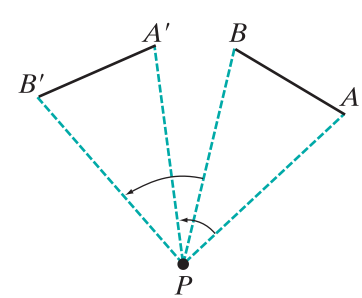
1. For A, a point that is not the fixed point P, if the image of A is A', then $PA = PA'$ and $m\angle APA' = d$.

2. The image of the center of rotation P is P.

In the figure, P is the center of rotation. If A is rotated about P to A', and B is rotated the same number of degrees to B', then $m\angle APA' = m\angle BPB'$. Since P is the center of rotation, $PA = PA'$ and $PB = PB'$, and

$$m\angle APA' - m\angle BPA' = m\angle APB$$

$$m\angle BPB' - m\angle BPA' = m\angle A'PB'$$



Therefore, $m\angle APB = m\angle A'PB'$ and $\triangle APB \cong \triangle A'PB'$ by SAS. Because corresponding parts of congruent triangles are congruent and equal in measure, $AB = A'B'$, that is, that distance is preserved under a rotation. We have just proved the following theorem:

Theorem 6.8 Distance is preserved under a rotation about a fixed point.

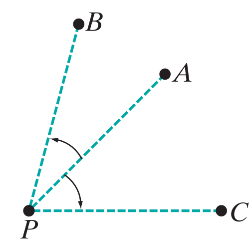
As we have seen when studying other transformations, when distance is preserved, angle measure, collinearity, and midpoint are also preserved.

► **Under a rotation about a fixed point, distance, angle measure, collinearity, and midpoint are preserved.**

We use $\boldsymbol{R_{P,d}}$ as a symbol for the image under a rotation of d degrees about point P. For example, the statement "$R_{O,30°}(A) = B$" can be read as "the image of A under a rotation of 30° degrees about the origin is B."

A rotation in the counterclockwise direction is called a **positive rotation**. For instance, B is the image of A under a rotation of 30° about P.

A rotation in the clockwise direction is called a **negative rotation**. For instance, C is the image of A under a rotation of −45° about P.



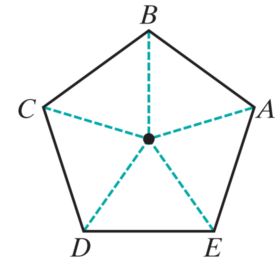
Rotational Symmetry

DEFINITION

A figure is said to have **rotational symmetry** if the figure is its own image under a rotation and the center of rotation is the only fixed point.

Many letters, as well as designs in the shapes of wheels, stars, and polygons, have *rotational symmetry*. When a figure has rotational symmetry under a rotation of $d°$, we can rotate the figure by $d°$ to an image that is an identical figure. Each figure shown below has rotational symmetry.

Any *regular polygon* (a polygon with all sides congruent and all angles congruent) has rotational symmetry. When regular pentagon $ABCDE$ is rotated $\frac{360°}{5}$, or 72°, about its center, the image of every point of the figure is a point of the figure. Under this rotation:



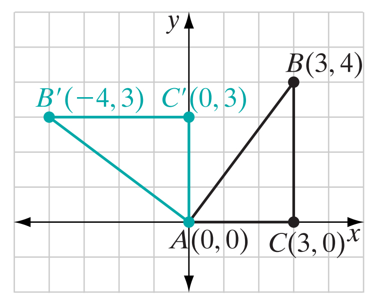
$A \to B \quad B \to C \quad C \to D \quad D \to E \quad E \to A$

The figure would also have rotational symmetry if it were rotated through a multiple of 72° (144°, 216°, or 288°). If it were rotated through 360°, every point would be its own image. Since this is true for every figure, we do not usually consider a 360° rotation as rotational symmetry.

Rotations in the Coordinate Plane

The most common rotation in the coordinate plane is a **quarter turn** about the origin, that is, a counterclockwise rotation of 90° about the origin.

In the diagram, the vertices of right triangle ABC are $A(0,0)$, $B(3, 4)$ and $C(3, 0)$. When rotated 90° about the origin, A remains fixed because it is the center of rotation. The image of C, which is on the x-axis and 3 units from the origin, is $C'(0, 3)$ on the y-axis and 3 units from the origin. Since $\overline{CB}$ is a vertical line 4 units long, its image is a horizontal line 4 units long and to the left of the y-axis. Therefore, the image of B is $B'(-4, 3)$. Notice that the x-coordinate of B' is the negative of the y-coordinate of B and the y-coordinate of B' is the x-coordinate of B.

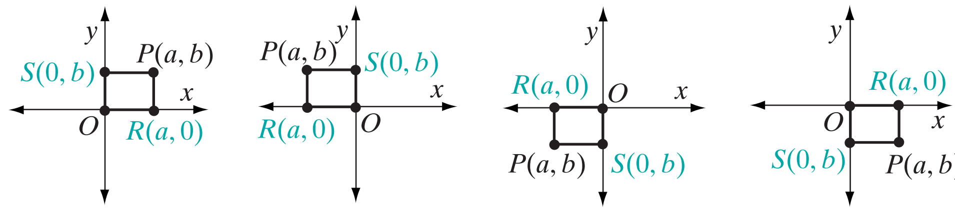


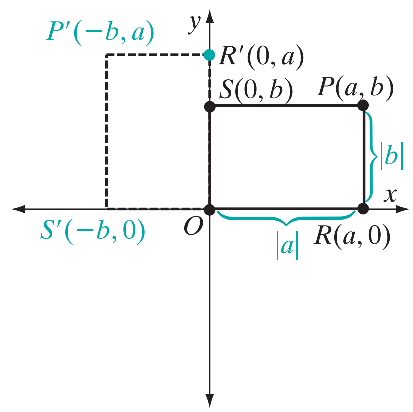
The point $B(3, 4)$ and its image $B'(-4, 3)$ in the above example suggest a rule for the coordinates of the image of any point $P(x, y)$ under a counterclockwise rotation of 90° about the origin.

Theorem 6.9

> Under a counterclockwise rotation of 90° about the origin, the image of $P(a, b)$ is $P'(-b, a)$.

Proof: We will prove this theorem by using a rectangle with opposite vertices at the origin and at P. Note that in quadrants I and II, when b is positive, $-b$ is negative, and in quadrants III and IV, when b is negative, $-b$ is positive.





Let $P(a, b)$ be any point not on an axis and $O(0, 0)$ be the origin. Let $R(a, 0)$ be the point on the x-axis on the same vertical line as P and $S(0, b)$ be the point on the y-axis on the same horizontal line as P. Therefore, $PR = |b - 0| = |b|$ and $PS = |a - 0| = |a|$.

Under a rotation of 90° about the origin, the image of $ORPS$ is $OR'P'S'$. Then, $OR = OR'$ and $m\angle R'OR = 90$. This means that since $\overline{OR}$ is a horizontal segment, $\overline{OR'}$ is a vertical segment with the same length. Therefore, since R is on the x-axis, R' is on the y-axis and the coordinates of R' are $(0, a)$. Similarly, since S is on the y-axis, S' is on the x-axis and the coordinates of S' are $(-b, 0)$. Point P' is on the same horizontal line as R' and therefore has the same y-coordinate as R', and P' is on the same vertical line as S' and has the same x-coordinate as S'. Therefore, the coordinates of P' are $(-b, a)$. ∎

The statement of Theorem 6.9 may be written as:

$$\mathbf{R_{O,90^\circ}(x, y) = (-y, x)} \quad \text{or} \quad \mathbf{R_{90^\circ}(x, y) = (-y, x)}$$

When the point that is the center of rotation is not named, it is understood to be O, the origin.

Note: The symbol R is used to designate both a point reflection and a rotation.

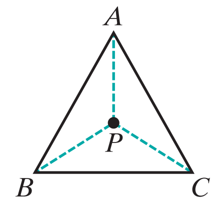
1. When the symbol R is followed by a letter that designates a point, it represents a reflection in that point.
2. When the symbol R is followed by both a letter that designates a point and the number of degrees, it represents a rotation of the given number of degrees about the given point.
3. When the symbol R is followed by the number of degrees, it represents a rotation of the given number of degrees about the origin.

EXAMPLE 1

Point P is at the center of equilateral triangle ABC (the point at which the perpendicular bisectors of the sides intersect) so that:

$$PA = PB = PC \quad \text{and} \quad m\angle APB = m\angle BPC = m\angle CPA$$

Under a rotation about P for which the image of A is B, find:



Answers

a. The number of degrees in the rotation. — **a.** $\frac{360°}{3} = 120°$

b. The image of B. — **b.** C

c. The image of $\overline{CA}$. — **c.** $\overline{AB}$

d. The image of $\angle CAB$. — **d.** $\angle ABC$

EXAMPLE 2

What are the coordinates of the image of $P(-2, 3)$ under $R_{90°}$?

Solution The image of (x, y) is $(-y, x)$. Therefore, the image of $(-2, 3)$ is $(-3, -2)$.

Answer $(-3, -2)$

Exercises

Writing About Mathematics

1. A point in the coordinate plane is rotated 180° about the origin by rotating the point counterclockwise 90° and then rotating the image counterclockwise 90°.

a. Choose several points in the coordinate plane and find their images under a rotation of 180°. What is the image of $P(x, y)$ under this rotation?

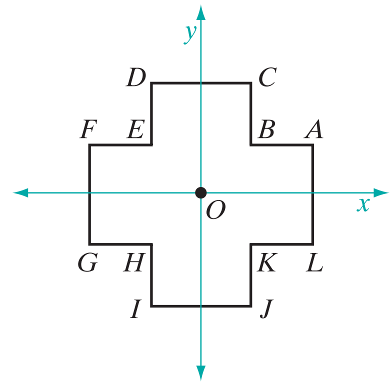
b. For what other transformation does $P(x, y)$ have the same image?

c. Is a 180° rotation about the origin equivalent to the transformation found in part **b**?

2. Let Q be the image of $P(x, y)$ under a clockwise rotation of 90° about the origin and R be the image of Q under a clockwise rotation of 90° about the origin. For what two different transformations is R the image of P?

Developing Skills

For 3 and 4, refer to the figure at the right.



3. What is the image of each of the given points under $R_{90°}$?

a. A **b.** B **c.** C **d.** G

e. H **f.** J **g.** K **h.** L

4. What is the image of each of the given points under $R_{-90°}$?

a. A **b.** B **c.** C **d.** G

e. H **f.** J **g.** K **h.** L

5. The vertices of rectangle $ABCD$ are $A(2, -1)$, $B(5, -1)$, $C(5, 4)$, and $D(2, 4)$.

a. What are the coordinates of the vertices of $A'B'C'D'$, the image of $ABCD$ under a counterclockwise rotation of 90° about the origin?

b. Is $A'B'C'D'$ a rectangle?

c. Find the coordinates of the midpoint of $\overline{AB}$ and of $\overline{A'B'}$. Is the midpoint of $\overline{A'B'}$ the image of the midpoint of $\overline{AB}$ under this rotation?

6. a. What are the coordinates of Q, the image of $P(1, 3)$ under a counterclockwise rotation of 90° about the origin?

b. What are the coordinates of R, the image of Q under a counterclockwise rotation of 90° about the origin?

c. What are the coordinates of S, the image of R under a counterclockwise rotation of 90° about the origin?

d. What are the coordinates of P', the image of P under a clockwise rotation of 90° about the origin?

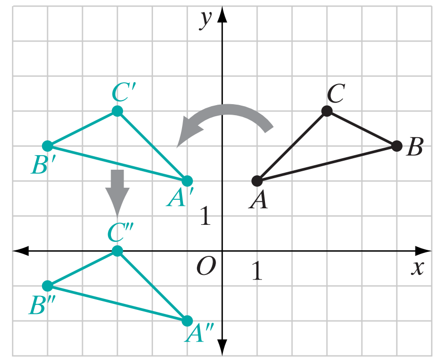
e. Explain why S and P' are the same point.

6-7 GLIDE REFLECTIONS

When two transformations are performed, one following the other, we have a **composition of transformations**. The first transformation produces an image and the second transformation is performed on that image. One such composition that occurs frequently is the composition of a line reflection and a translation.

DEFINITION

A **glide reflection** is a composition of transformations of the plane that consists of a line reflection and a translation in the direction of the line of reflection performed in either order.



The vertices of $\triangle ABC$ are $A(1, 2)$, $B(5, 3)$, and $C(3, 4)$. Under a reflection in the y-axis, the image of $\triangle ABC$ is $\triangle A'B'C'$ whose vertices are $A'(-1, 2)$, $B'(-5, 3)$, and $C'(-3, 4)$. Under the translation $T_{0,-4}$, the image of $\triangle A'B'C'$ is $\triangle A''B''C''$ whose vertices are $A''(-1, -2)$, $B''(-5, -1)$, and $C''(-3, 0)$. Under a glide reflection, the image of $\triangle ABC$ is $\triangle A''B''C''$. Note that the line of reflection, the y-axis, is a vertical line. The translation is in the vertical direction because the x-coordinate of the translation is 0. Distance is preserved under a line reflection and under a translation: $\triangle ABC \cong \triangle A'B'C' \cong \triangle A''B''C''$. Distance is preserved under a glide reflection. We have just proved the following theorem:

Theorem 6.10

Under a glide reflection, distance is preserved.

As we have seen with other transformations, when distance is preserved, angle measure, collinearity, and midpoint are also preserved. Therefore, we may make the following statement:

► **Under a glide reflection, distance, angle measure, collinearity, and midpoint are preserved.**

In this chapter, we have studied five transformations: line reflection, point reflection, translation, rotation, and glide reflection. Each of these transformations is called an *isometry* because it preserves distance.

DEFINITION

An **isometry** is a transformation that preserves distance.

EXAMPLE 1

The vertices of $\triangle PQR$ are $P(2, 1)$, $Q(4, 1)$, and $R(4, 3)$.

a. Find $\triangle P'Q'R'$, the image of $\triangle PQR$ under $r_{y=x}$ followed by $T_{-3,-3}$.

b. Find $\triangle P''Q''R''$, the image of $\triangle PQR$ under $T_{-3,-3}$ followed by $r_{y=x}$.

c. Are $\triangle P'Q'R'$ and $\triangle P''Q''R''$ the same triangle?

d. Are $r_{y=x}$ followed by $T_{-3,-3}$ and $T_{-3,-3}$ followed by $r_{y=x}$ the same glide reflection? Explain.

e. Write a rule for this glide reflection.

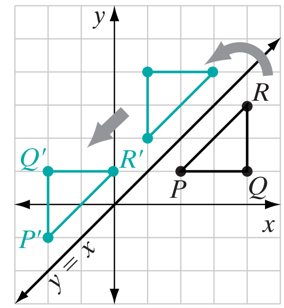
Solution **a.** $r_{y=x}(x, y) = (y, x)$ $T_{-3,-3}(y, x) = (y - 3, x - 3)$

$r_{y=x}(2, 1) = (1, 2)$ $T_{-3,-3}(1, 2) = (-2, -1)$

$r_{y=x}(4, 1) = (1, 4)$ $T_{-3,-3}(1, 4) = (-2, 1)$

$r_{y=x}(4, 3) = (3, 4)$ $T_{-3,-3}(3, 4) = (0, 1)$

The vertices of $\triangle P'Q'R'$ are $P'(-2, -1)$, $Q'(-2, 1)$, and $R'(0, 1)$. *Answer*



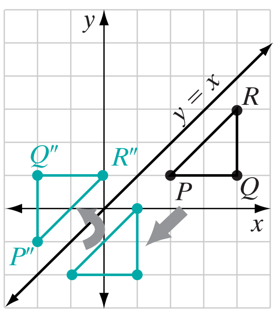
b. $T_{-3,-3}(x, y) = (x - 3, y - 3)$ $r_{y=x}(x - 3, y - 3) = (y - 3, x - 3)$

$T_{-3,-3}(2, 1) = (-1, -2)$ $r_{y=x}(-1, -2) = (-2, -1)$

$T_{-3,-3}(4, 1) = (1, -2)$ $r_{y=x}(1, -2) = (-2, 1)$

$T_{-3,-3}(4, 3) = (1, 0)$ $r_{y=x}(1, 0) = (0, 1)$

The vertices of $\triangle P''Q''R''$ are $P''(-2, -1)$, $Q''(-2, 1)$, and $R''(0, 1)$. *Answer*



c. $\triangle P'Q'R'$ and $\triangle P''Q''R''$ are the same triangle. *Answer*

d. The image of (x, y) under $r_{y=x}$ followed by $T_{-3,-3}$ is

$$(x, y) \rightarrow (y, x) \rightarrow (y - 3, x - 3)$$

The image of (x, y) under $T_{-3,-3}$ followed by $r_{y=x}$ is

$$(x, y) \rightarrow (x - 3, y - 3) \rightarrow (y - 3, x - 3)$$

$r_{y=x}$ followed by $T_{-3,-3}$ and $T_{-3,-3}$ followed by $r_{y=x}$ are the same glide reflection. *Answer*

e. $(x, y) \rightarrow (y - 3, x - 3)$ *Answer*

EXAMPLE 2

Is a reflection in the y-axis followed by the translation $T_{5,5}$ a glide reflection?

Solution The y-axis is a vertical line. The translation $T_{5,5}$ is not a translation in the vertical direction. Therefore, a reflection in the y-axis followed by the translation $T_{5,5}$ is not a glide reflection.

Exercises

Writing About Mathematics

1. Does a glide reflection have any fixed points? Justify your answer.

2. In a glide reflection, the line reflection and the translation can be done in either order. Is the composition of any line reflection and any translation always the same in either order? Justify your answer.

Developing Skills

In 3–8, **a.** find the coordinates of the image of the triangle whose vertices are $A(1, 1)$, $B(5, 4)$, and $C(3, 5)$ under the given composition of transformations. **b.** Sketch $\triangle ABC$ and its image. **c.** Explain why the given composition of transformations is or is not a glide reflection. **d.** Write the coordinates of the image of (a, b) under the given composition of transformations.

3. A reflection in the x-axis followed by a translation of 4 units to the right.

4. A reflection in the y-axis followed by a translation of 6 units down.

5. $T_{5,0}$ followed by $r_{x\text{-axis}}$

6. $T_{3,3}$ followed by $r_{y\text{-axis}}$

7. $T_{3,3}$ followed by $r_{y=x}$

8. R_{90} followed by $T_{1,2}$

Applying Skills

9. The point $A'(-4, 5)$ is the image of A under a glide reflection that consists of a reflection in the x-axis followed by the translation $T_{-2,0}$. What are the coordinates of A?

10. Under a glide reflection, the image of $A(2, 4)$ is $A'(-2, -7)$ and the image of $B(3, 7)$ is $B'(-3, -4)$.

a. If the line of reflection is a vertical line, write an equation for the line reflection and a rule for the translation.

b. What are the coordinates of C', the image of $C(0, 7)$, under the same glide reflection?

11. Under a glide reflection, the image of $A(3, 4)$ is $A'(-1, -4)$ and the image of $B(5, 5)$ is $B'(1, -5)$.

a. If the line of reflection is a horizontal line, write a rule for the line reflection and a rule for the translation.

b. What are the coordinates of C', the image of $C(4, 2)$, under the same glide reflection?

12. **a.** For what transformation or transformations are there no fixed points?

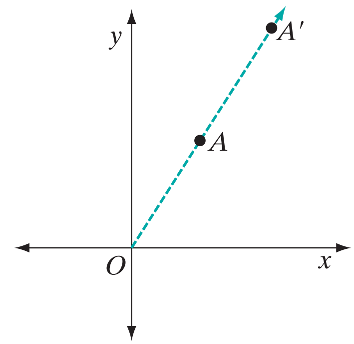
b. For what transformation or transformations is there exactly one fixed point?

c. For what transformation or transformations are there infinitely many fixed points?

6-8 DILATIONS IN THE COORDINATE PLANE

In this chapter, we have learned about transformations in the plane that are isometries, that is, transformations that preserve distance. There is another transformation in the plane that preserves angle measure but not distance. This transformation is a *dilation*.

For example, in the coordinate plane, a dilation of 2 with center at the origin will stretch each ray by a factor of 2. If the image of A is A', then A' is a point on $\overrightarrow{OA}$ and $OA' = 2OA$.



DEFINITION

A **dilation** of k is a transformation of the plane such that:

1. The image of point O, the center of dilation, is O.
2. When k is positive and the image of P is P', then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the same ray and $OP' = kOP$.
3. When k is negative and the image of P is P', then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are opposite rays and $OP' = -kOP$.

Note: In step 3, when k is negative, $-k$ is positive.

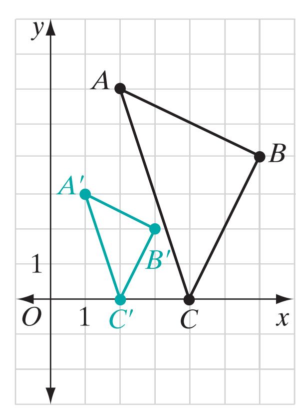
In the coordinate plane, the center of dilation is usually the origin. If the center of dilation is not the origin, the coordinates of the center will be given.

In the coordinate plane, under a dilation of k with the center at the origin:

$$P(x, y) \rightarrow P'(kx, ky) \quad \text{or} \quad D_k(x, y) = (kx, ky)$$

For example, the image of $\triangle ABC$ is $\triangle A'B'C'$ under a dilation of $\frac{1}{2}$. The vertices of $\triangle ABC$ are $A(2, 6)$, $B(6, 4)$, and $C(4, 0)$. Under a dilation of $\frac{1}{2}$, the rule is

$$D_{\frac{1}{2}}(x, y) = \left(\tfrac{1}{2}x, \tfrac{1}{2}y\right)$$
$$A(2, 6) \rightarrow A'(1, 3)$$
$$B(6, 4) \rightarrow B'(3, 2)$$
$$C(4, 0) \rightarrow C'(2, 0)$$



By the definition of a dilation, distance is not preserved. We will prove in Chapter 12 that angle measure, collinearity, and midpoint are preserved.

► **Under a dilation about a fixed point, distance is *not* preserved.**

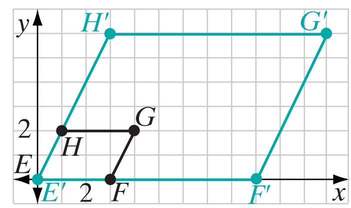
EXAMPLE 1

The coordinates of parallelogram $EFGH$ are $E(0, 0)$, $F(3, 0)$, $G(4, 2)$, and $H(1, 2)$. Under D_3, the image of $EFGH$ is $E'F'G'H'$.

a. Find the coordinates of the vertices of $E'F'G'H'$.

b. Let M be the midpoint of $\overline{EF}$. Find the coordinates of M and of M', the image of M. Verify that the midpoint is preserved.

Solution **a.** $D_3(x, y) = (3x, 3y)$. Therefore, $E'(0, 0)$, $F'(9, 0)$, $G'(12, 6)$, and $H'(3, 6)$. *Answer*



b. Since E and F lie on the x-axis, EF is the absolute value of the difference of their x-coordinates. $EF = |3 - 0| = 3$. Therefore, the midpoint, M, is $1\frac{1}{2}$ units from E or from F on the x-axis. The coordinates of M are $\left(0 + 1\frac{1}{2}, 0\right) = \left(1\frac{1}{2}, 0\right)$.

Since E' and F' lie on the x-axis, $E'F'$ is the absolute value of the difference of their x-coordinates. $E'F' = |9 - 0| = 9$. Therefore, the midpoint, M', is $4\frac{1}{2}$ units from E' or from F' on the x-axis. The coordinates of M are $\left(0 + 4\frac{1}{2}, 0\right) = \left(4\frac{1}{2}, 0\right)$.

$D_3\left(1\frac{1}{2}, 0\right) = \left(4\frac{1}{2}, 0\right)$. Therefore, the image of M is M' and midpoint is preserved. *Answer*

EXAMPLE 2

Find the coordinates of P', the image of $P(4, -5)$ under the composition of transformations: D_2 followed by $r_{x\text{-axis}}$.

Solution The dilation is to be performed first: $D_2(4, -5) = (8, -10)$. Then perform the reflection, using the result of the dilation: $r_{x\text{-axis}}(8, -10) = (8, 10)$.

Answer $P' = (8, 10)$

Exercises

Writing About Mathematics

1. Let $\triangle A'B'C'$ be the image of $\triangle ABC$ under a dilation of k. When $k > 1$, how do the lengths of the sides of $\triangle A'B'C'$ compare with the lengths of the corresponding sides of $\triangle ABC$? Is a dilation an isometry?

2. Let $\triangle A'B'C'$ be the image of $\triangle ABC$ under a dilation of k. When $0 < k < 1$, how do the lengths of the sides of $\triangle A'B'C'$ compare with the lengths of the corresponding sides of $\triangle ABC$?

Developing Skills

In 3–6, use the rule $(x, y) \rightarrow (4x, 4y)$ to find the coordinates of the image of each given point.

3. $(3, 7)$ **4.** $(-4, 2)$

5. $(2, 0)$ **6.** $(-1, 9)$

In 7–10, find the coordinates of the image of each given point under D_5.

7. $(2, 2)$ **8.** $(1, 10)$

9. $(-3, 5)$ **10.** $(0, 4)$

In 11–14, each given point is the image under D_2. Find the coordinates of each preimage.

11. $(6, -2)$ **12.** $(4, 0)$

13. $(-6, -5)$ **14.** $(10, 7)$

In 15–20, find the coordinates of the image of each given point under the given composition of transformations.

15. $D_4(1, 5)$ followed by $r_{y\text{-axis}}$ **16.** $D_{-2}(-3, 2)$ followed by R_{90°

17. $T_{2,1}(4, -2)$ followed by $D_{\frac{1}{2}}$ **18.** $D_3(5, -1)$ followed by $r_{x\text{-axis}}$

19. $T_{1,-1}(-4, 2)$ followed by D_{-2} **20.** $D_2(6, 3)$ followed by $r_{y=x}$

In 21–24, each transformation is the composition of a dilation and a reflection in either the x-axis or the y-axis. In each case, write a rule for composition of transformations for which the image of A is A'.

21. $A(2, 5) \rightarrow A'(4, -10)$ **22.** $A(3, -1) \rightarrow A'(-21, -7)$

23. $A(10, 4) \rightarrow A'(5, -2)$ **24.** $A(-20, 8) \rightarrow A'(5, 2)$

Applying Skills

25. The vertices of rectangle $ABCD$ are $A(1, -1)$, $B(3, -1)$, $C(3, 3)$, and $D(1, 3)$.

a. Find the coordinates of the vertices of $A'B'C'D'$, the image of $ABCD$ under D_4.

b. Show that $A'B'C'D'$ is a rectangle.

26. Show that when $k > 0$, a dilation of k with center at the origin followed by a reflection in the origin is the same as a dilation of $-k$ with center at the origin.

27. **a.** Draw $\triangle ABC$, whose vertices are $A(2, 3)$, $B(4, 3)$, and $C(4, 6)$.

b. Using the same set of axes, graph $\triangle A'B'C'$, the image of $\triangle ABC$ under a dilation of 3.

c. Using $\triangle ABC$ and its image $\triangle A'B'C'$, show that distance is *not* preserved under the dilation.

Hands-On Activity

In this activity, we will verify that angle measure, collinearity, and midpoint are preserved under a dilation.

Using geometry software, or a pencil, ruler and a protractor, draw the pentagon $A(1, 2)$, $B(4, 2)$, $C(6, 4)$, $D(2, 8)$, $E(1, 6)$, and point $P(4, 6)$ on $\overline{CD}$. For each given dilation: **a.** Measure the corresponding angles. Do they appear to be congruent? **b.** Does the image of P appear to be on the image of $\overline{CD}$? **c.** Does the image of the midpoint of $\overline{AE}$ appear to be the midpoint of the image of $\overline{AE}$?

(1) D_3 (3) $D_{\frac{1}{2}}$

(2) D_{-3} (4) $D_{-\frac{1}{2}}$

6-9 TRANSFORMATIONS AS FUNCTIONS

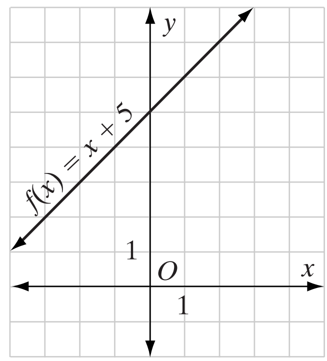
A **function** is a set of ordered pairs in which no two pairs have the same first element. For example, the equation $y = x + 5$ describes the set of ordered pairs of real numbers in which the second element, y, is 5 more than the first element, x. The set of first elements is the **domain** of the function and the set of second elements is the **range**. Some pairs of this function are $(-3, 2)$, $(-2.7, 2.3)$, $\left(\frac{2}{3}, \frac{17}{3}\right)$, $(0, 5)$, and $(1, 6)$. Since the set of real numbers is infinite, the set of pairs of this function cannot be listed completely and are more correctly defined when they are described. Each of the following notations describes the function in which the second element is 5 more than the first.

$$f = \{(x, y) \mid y = x + 5\} \qquad f\colon x \rightarrow x + 5 \qquad y = x + 5 \qquad f(x) = x + 5$$

Note that $f(x)$ and y each name the second element of the ordered pair.

Each of the transformations defined in this chapter is a one-to-one function. It is a set of ordered pairs in which the first element of each pair is a point of the plane and the second element is the image of that point under a transformation. For each point in the plane, there is one and only one image. For example, compare the function $f(x) = x + 5$ and a line reflection.

f: A one-to-one algebraic function
$f(x) = x + 5$



1. For every x in the domain there is one and only one y in the range. For example:

$$f(3) = 8$$
$$f(0) = 5$$
$$f(-2) = 3$$

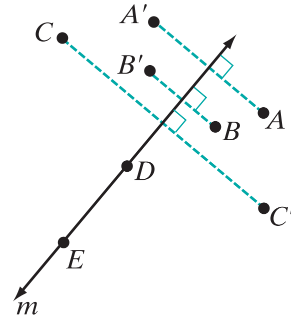
2. Every $f(x)$ or y in the range corresponds to one and only one value of x in the range. For example:

If $f(x) = 4$, then $x = -1$.
If $f(x) = 9$, then $x = 4$.

3. For the function f:

domain = {real numbers}
range = {real numbers}

r_m: A reflection in line m



1. For every point in the plane, there is one and only one image of that point in the plane. For example:

$$r_m(A) = A'$$
$$r_m(C) = C'$$
$$r_m(D) = D$$

2. Every point is the image of one and only one preimage. For example:
The point B' has the preimage B.
The point E has itself as the preimage.
The point C' has the preimage C.

3. For the function r_m:

domain = {points in the plane}
range = {points in the plane}

Composition of Transformations

We defined the composition of transformations as a combination of two transformations in which the first transformation produces an image and the second transformation is performed on that image.

For example, to indicate that A' is the image of A under a reflection in the line $y = x$ followed by the translation $T_{2,0}$, we can write

$$T_{2,0}(r_{y=x}(A)) = A'.$$

If the coordinates of A are $(2, 5)$, we start with A and perform the transformations from right to left as indicated by the parentheses, evaluating what is in parentheses first.

$$r_{y=x}(2, 5) = (5, 2) \quad \text{followed by} \quad T_{2,0}(5, 2) = (7, 2)$$

or

$$\begin{aligned} T_{2,0}(r_{y=x}(2, 5)) &= T_{2,0}(5, 2) \\ &= (7, 2) \end{aligned}$$

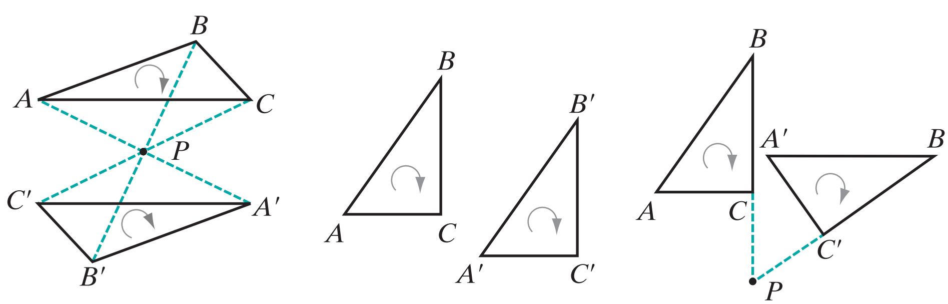
A small raised circle is another way of indicating a composition of transformations. $T_{2,0}(r_{y=x}(A)) = A'$ can also be written as

$$T_{2,0} \circ r_{y=x}(A) = A'.$$

Again, we start with the coordinates of A and move from right to left. Find the image of A under the reflection and then, using the coordinates of that image, find the coordinates under the translation.

Orientation

The figures below show the images of $\triangle ABC$ under a point reflection, a translation, and a rotation. In each figure, the vertices, when traced from A to B to C are in the clockwise direction, called the orientation of the points. The vertices of the images, when traced in the same order, from A' to B' to C' are also in the clockwise direction. Therefore, we say that under a point reflection, a translation, or a rotation, **orientation** is unchanged.



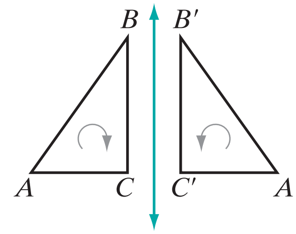
Point Reflection *Translation* *Rotation*

DEFINITION

A **direct isometry** is a transformation that preserves distance and orientation.

Point reflection, rotation, and translation are direct isometries.

The figure at the right shows $\triangle ABC$ and its image under a line reflection. In this figure, the vertices, when traced from A to B to C, are again in the clockwise direction. The vertices of the images, when traced in the same order, from A' to B' to C', are in the counterclockwise direction. Therefore, under a line reflection, orientation is changed or reversed.



Line Reflection

DEFINITION

An **opposite isometry** is a transformation that preserves distance but changes the order or orientation from clockwise to counterclockwise or from counterclockwise to clockwise.

A line reflection is an opposite isometry.

EXAMPLE 1

The vertices of $\triangle RST$ are $R(1,1)$, $S(6, 3)$, and $T(2, 5)$.

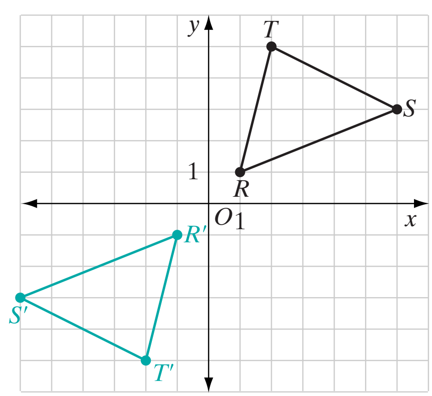
a. Sketch $\triangle RST$.

b. Find the coordinates of the vertices of $\triangle R'S'T'$ under the composition $r_{x\text{-axis}} \circ r_{y\text{-axis}}$ and sketch $\triangle R'S'T'$.

c. Is the composition a direct isometry?

d. For what single transformation is the image the same as that of $r_{x\text{-axis}} \circ r_{y\text{-axis}}$?

Solution **a.**



b. $r_{x\text{-axis}} \circ r_{y\text{-axis}}(1, 1) = r_{x\text{-axis}}(-1, 1)$
$= (-1, -1)$

$r_{x\text{-axis}} \circ r_{y\text{-axis}}(6, 3) = r_{x\text{-axis}}(-6, 3)$
$= (-6, -3)$

$r_{x\text{-axis}} \circ r_{y\text{-axis}}(2, 5) = r_{x\text{-axis}}(-2, 5)$
$= (-2, -5)$

c. A line reflection is an opposite isometry. The composition of two opposite isometries is a direct isometry. This composition is a direct isometry.

d. $r_{x\text{-axis}} \circ r_{y\text{-axis}}(x, y) = (-x, -y)$ and $R_O(x, y) = (-x, -y)$. The composition of a reflection in the y-axis followed by a reflection in the x-axis is a reflection in the origin. ❑

EXAMPLE 2

The transformation $(x, y) \to (-y + 3, x - 2)$ is the composition of what two transformations?

Solution Under a rotation of 90° about the origin, $(x, y) \to (-y, x)$.

When this rotation is followed by the translation $T_{3,-2}(-y, x) = (-y + 3, x - 2)$.

Answer $T_{3,-2} \circ R_{90°}(x,y) = T_{3,-2}(-y, x) = (-y + 3, x - 2)$

This solution is not unique.

Alternative Solution Under the translation $T_{-2,-3}(x, y) = (x - 2, y - 3)$.

When this translation is followed by the rotation $R_{90°}(x - 2, y - 3) = (-(y - 3), x - 2) = (-y + 3, x - 2)$.

Answer $R_{90°} \circ T_{-2,-3}(x, y) = R_{90°}(x - 2, y - 3) = (-y + 3, x - 2)$

Exercises

Writing About Mathematics

1. Owen said that since a reflection in the x-axis followed by a reflection in the y-axis has the same result as a reflection in the y-axis followed by a reflection in the x-axis, the composition of line reflections is a commutative operation. Do you agree with Owen? Justify your answer.

2. Tyler said that the composition of an even number of opposite isometries is a direct isometry and that the composition of an odd number of opposite isometries is an opposite isometry. Do you agree with Tyler? Justify your answer.

Developing Skills

In 3–11, find the image of $A(3, -2)$ under the given composition of transformations.

3. $r_{x\text{-axis}} \circ r_{y\text{-axis}}$
4. $r_{y\text{-axis}} \circ r_{x\text{-axis}}$
5. $R_O \circ r_{x\text{-axis}}$
6. $R_{90°} \circ R_O$
7. $T_{-1,4} \circ r_{y=x}$
8. $R_{90°} \circ R_{90°}$
9. $r_{y\text{-axis}} \circ r_{y=x}$
10. $R_O \circ T_{3,-2}$
11. $T_{-2,4} \circ D_{-2}$
12. A reflection in the line $y = x$ followed by a rotation of 90° about the origin is equivalent to what single transformation?
13. A rotation of 90° about the origin followed by another rotation of 90° about the origin is equivalent to what single transformation?

14. The vertices of $\triangle DEF$ are $D(3, 2)$, $E(5, 5)$, and $F(4, -1)$.

a. If $T_{-3,0} \circ r_{x\text{-axis}}(\triangle DEF) = \triangle D'E'F'$, find the coordinates of the vertices of $\triangle D'E'F'$.

b. The composition $T_{-3,0} \circ r_{x\text{-axis}}$ is an example of what type of transformation?

c. Is the composition $T_{-3,0} \circ r_{x\text{-axis}}$ a direct or an opposite isometry?

Applying Skills

15. Prove that under a reflection in the line $y = -x$, $A(a, b) \rightarrow A'(-b, -a)$.

Suggested plan: Let $B(a, -a)$ and $C(-b, b)$ be points on the line $y = -x$. Show that the line $y = -x$ is the perpendicular bisector of $\overline{AA'}$.

16. Prove that the composition of a reflection in the line $y = -x$ followed by a reflection in the line $y = x$ is a reflection in the origin.

17. Prove that the composition of a reflection in the line $y = x$ followed by a reflection in the line $y = -x$ is a reflection in the origin.

CHAPTER SUMMARY

Definitions to Know

- A **transformation** is a one-to-one correspondence between two sets of points, S and S', when every point in S corresponds to one and only one point in S' called its **image**, and every point in S' is the image of one and only one point in S' called its **preimage**.
- A **reflection in line *k*** is a transformation in a plane such that:
  1. If point A is not on k, then the image of A is A' where k is the perpendicular bisector of $\overline{AA'}$.
  2. If point A is on k, the image of A is A.
- A figure has **line symmetry** when the figure is its own image under a line reflection.
- A **point reflection in *P*** is a transformation plane such that:
  1. If point A is not point P, then the image of A is A' where P the midpoint of $\overline{AA'}$.
  2. The point P is its own image.
- A **translation** is a transformation in a plane that moves every point in the plane the same distance in the same direction.
- A **translation of *a* units in the horizontal direction and *b* units in the vertical direction** is a transformation in a plane such that the image of $P(x, y)$ is $P'(x + a, y + b)$.

- A **rotation** is a transformation of a plane about a fixed point P through an angle of d degrees such that:
  1. For A, a point that is not the fixed point, if the image of A is A', then $PA = PA'$ and $m\angle APA' = d$.
  2. The image of the center of rotation P is P.
- A **quarter turn** is a counterclockwise rotation of 90°.
- A **composition of transformations** is a combination of two transformations in which the first transformation produces an image and the second transformation is performed on that image.
- A **glide reflection** is a composition of transformations of the plane that consists of a line reflection and a translation in the direction of the line of reflection in either order.
- An **isometry** is a transformation that preserves distance.
- A **dilation** of k is a transformation of the plane such that:
  1. The image of point O, the center of dilation, is O.
  2. When k is positive and the image of P is P', then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the same ray and $OP' = kOP$.
  3. When k is negative and the image of P is P', then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are opposite rays and $OP' = -kOP$.
- A **function** is a set of ordered pairs in which no two pairs have the same first element.
- The set of first elements is the **domain** of the function.
- The set of second elements is the **range** of the function.
- A **direct isometry** is a transformation that preserves distance and orientation.
- An **opposite isometry** is a transformation that preserves distance and reverses orientation.

Postulates

6.1 Two points are on the same horizontal line if and only if they have the same y-coordinates.
6.2 The length of a horizontal line segment is the absolute value of the difference of the x-coordinates.
6.3 Two points are on the same vertical line if and only if they have the same x-coordinate.
6.4 The length of a vertical line segment is the absolute value of the difference of the y-coordinates.
6.5 Each vertical line is perpendicular to each horizontal line.

Theorems and Corollaries

6.1 Under a line reflection, distance is preserved.
6.1a Under a reflection, angle measure is preserved.
6.1b Under a reflection, collinearity is preserved.
6.1c Under a reflection, midpoint is preserved.

6.2 Under a reflection in the y-axis, the image of $P(a, b)$ is $P'(-a, b)$.
6.3 Under a reflection in the x-axis, the image of $P(a, b)$ is $P'(a, -b)$.
6.4 Under a reflection in the line $y = x$, the image of $P(a, b)$ is $P'(b, a)$.
6.5 Under a point reflection, distance is preserved.
6.5a Under a point reflection, angle measure is preserved.
6.5b Under a point reflection, collinearity is preserved.
6.5c Under a point reflection, midpoint is preserved.
6.6 Under a reflection in the origin, the image of $P(a, b)$ is $P'(-a, -b)$.
6.7 Under a translation, distance is preserved.
6.7a Under a translation, angle measure is preserved.
6.7b Under a translation, collinearity is preserved.
6.7c Under a translation, midpoint is preserved.
6.8 Distance is preserved under a rotation about a fixed point.
6.9 Under a counterclockwise rotation of 90° about the origin, the image of $P(a, b)$ is $P'(-b, a)$.
6.10 Under a glide reflection, distance is preserved.

VOCABULARY

6-1 Coordinate plane • x-axis • y-axis • Origin • Coordinates • Ordered pair • x-coordinate (abscissa) • y-coordinate (ordinate) • (x, y)

6-2 Line of reflection • Line reflection • Fixed point • Transformation • Preimage • Image • Reflection in line k • r_k • Axis of symmetry • Line symmetry

6-3 $r_{y\text{-axis}}$ • $r_{x\text{-axis}}$ • $r_{y=x}$

6-4 Point reflection in P • R_P • Point symmetry • R_O

6-5 Translation • Translation of a units in the horizontal direction and b units in the vertical direction • $T_{a,b}$ • Translational symmetry

6-6 Rotation • $R_{P,d}$ • Positive rotation • Negative rotation • Rotational symmetry • Quarter turn

6-7 Composition of transformations • Glide reflection • Isometry

6-8 Dilation • D_k

6-9 Function • Domain • Range • Orientation • Direct isometry • Opposite Isometry

REVIEW EXERCISES

1. Write the equation of the vertical line that is 2 units to the left of the origin.

2. Write the equation of the horizontal line that is 4 units below the origin.

3. a. On graph paper, draw the polygon $ABCD$ whose vertices are $A(-4, 0)$, $B(0, 0)$, $C(3, 3)$, and $D(-4, 3)$.

b. Find the area of polygon $ABCD$.

In 4–11: **a.** Find the image of $P(5, -3)$ under each of the given transformations. **b.** Name a fixed point of the transformation if one exists.

4. $r_{x\text{-axis}}$ **5.** R_O **6.** $R_{90°}$ **7.** $T_{0,3}$

8. $r_{x\text{-axis}} \circ T_{2,2}$ **9.** $r_{x\text{-axis}} \circ r_{y=x}$ **10.** $r_{y=x} \circ r_{x\text{-axis}}$ **11.** $R_{(2,2)} \circ T_{5,-3}$

12. Draw a quadrilateral that has exactly one line of symmetry.

13. Draw a quadrilateral that has exactly four lines of symmetry.

14. Print a letter that has rotational symmetry.

15. The letters **S** and **N** have point symmetry. Print another letter that has point symmetry.

16. What transformations are opposite isometries?

17. What transformation is not an isometry?

18. a. On graph paper, locate the points $A(3, 2)$, $B(3, 7)$, and $C(-2, 7)$. Draw $\triangle ABC$.

b. Draw $\triangle A'B'C'$, the image of $\triangle ABC$ under a reflection in the origin, and write the coordinates of its vertices.

c. Draw $\triangle A''B''C''$, the image of $\triangle A'B'C'$ under a reflection in the y-axis, and write the coordinates of its vertices.

d. Under what single transformation is $\triangle A''B''C''$ the image of $\triangle ABC$?

19. a. On graph paper, locate the points $R(-4, -1)$, $S(-1, -1)$, and $T(-1, 2)$. Draw $\triangle RST$.

b. Draw $\triangle R'S'T'$, the image of $\triangle RST$ under a reflection in the origin, and write the coordinates of its vertices.

c. Draw $\triangle R''S''T''$, the image of $\triangle R'S'T'$ under a reflection in the line whose equation is $y = 4$, and write the coordinates of its vertices.

20. a. Write the coordinates of the image, under the correspondence $(x, y) \rightarrow (x, 0)$, of each of the following ordered pairs:

$$(3, 5), (3, 3), (-1, -1), (-1, 5).$$

b. Explain why $(x, y) \rightarrow (x, 0)$ is not a transformation.

21. The vertices of $\triangle MAT$ have coordinates $M(1, 3)$, $A(2, 2)$, and $T(-2, 2)$.

a. Find $\triangle M'A'T'$, the image of $\triangle MAT$ under the composition $r_{x\text{-axis}} \circ D_3$.

b. Find $\triangle M''A''T''$, the image of $\triangle MAT$ under the composition $D_3 \circ r_{x\text{-axis}}$.

c. Are $r_{x\text{-axis}} \circ D_3$ and $D_3 \circ r_{x\text{-axis}}$ equivalent transformations? Justify your answer.

Exploration

Designs for wallpaper, wrapping paper, or fabric often repeat the same pattern in different arrangements. Such designs are called **tessellations**. Find examples of the use of line reflections, point reflections, translations and glide reflections in these designs.

CUMULATIVE REVIEW Chapters 1–6

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The product $-7a(3a - 1)$ can be written as
(1) $-21a - 7a$ (2) $-21a + 7a$ (3) $-21a^2 + 7a$ (4) $-21a^2 - 1$

2. In the coordinate plane, the point whose coordinates are $(-2, 0)$ is
(1) on the x-axis
(2) on the y-axis
(3) in the first quadrant
(4) in the fourth quadrant

3. If D is not on $\overline{ABC}$, then
(1) $AD + DC = AC$
(2) $AD + DC > AC$
(3) $AD + DC < AC$
(4) $AB + BC > AD + DC$

4. If $\overleftrightarrow{ABC}$ is the perpendicular bisector of $\overline{DBE}$, which of the following could be *false*?
(1) $AD = AE$ (2) $DB = BE$ (3) $CD = CE$ (4) $AB = BC$

5. $\triangle DEF$ is not congruent to $\triangle LMN$, $DE = LM$, and $EF = MN$. Which of the following must be true?
(1) $\triangle DEF$ and $\triangle LMN$ are not both right triangles.
(2) $m\angle D \neq m\angle L$
(3) $m\angle F \neq m\angle N$
(4) $m\angle E \neq m\angle M$

6. Under a reflection in the origin, the image of $(-2, 4)$ is
(1) $(-2, -4)$ (2) $(2, 4)$ (3) $(2, -4)$ (4) $(4, -2)$

7. If $PQ = RQ$, then which of the following must be true?
(1) Q is the midpoint of $\overline{PR}$.
(2) Q is on the perpendicular bisector of $\overline{PR}$.
(3) $PQ + QR = PR$
(4) Q is between P and R.

8. Which of the following always has a line of symmetry?
(1) a right triangle
(2) a scalene triangle
(3) an isosceles triangle
(4) an acute triangle

9. In the coordinate plane, two points lie on the same vertical line. Which of the following must be true?
(1) The points have the same x-coordinates.
(2) The points have the same y-coordinates.
(3) The points lie on the y-axis.
(4) The points lie on the x-axis.

10. Angle A and angle B are complementary angles. If $m\angle A = x + 42$ and $m\angle B = 2x - 12$, the measure of the smaller angle is
(1) 20 (2) 28 (3) 62 (4) 88

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The image of $\triangle ABC$ under a reflection in the line $y = x$ is $\triangle ADE$. If the coordinates of A are $(-2, b)$, what is the value of b? Explain your answer.

12. What are the coordinates of the midpoint of a line segment whose endpoints are $A(2, -5)$ and $B(2, 3)$?

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. In quadrilateral $ABCD$, prove that $\angle ABC \cong \angle ADC$ if $\overline{AEC}$ is the perpendicular bisector of $\overline{BED}$.

14. The measure of $\angle R$ is 12 degrees more than three times the measure of $\angle S$. If $\angle S$ and $\angle R$ are supplementary angles, find the measure of each angle.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. The measures of the angles of $\triangle ABC$ are unequal ($m\angle A \neq m\angle B$, $m\angle B \neq m\angle C$, and $m\angle A \neq m\angle C$). Prove that $\triangle ABC$ is a scalene triangle.

16. *Given:* $T_{3,-2}(\triangle ABC) = \triangle A'B'C'$ and $\triangle XYZ$ with $XY = A'B'$, $YZ = B'C'$, and $\angle Y \cong \angle B$.

Prove: $\triangle ABC \cong \triangle XYZ$