

CHAPTER

5

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CONGRUENCE BASED ON TRIANGLES

The SSS postulate tells us that a triangle with sides of given lengths can have only one size and shape. Therefore, the area of the triangle is determined. We know that the area of a triangle is one-half the product of the lengths of one side and the altitude to that side. But can the area of a triangle be found using only the lengths of the sides? A formula to do this was known by mathematicians of India about 3200 B.C. In the Western world, Heron of Alexandria, who lived around 75 B.C., provided in his book *Metrica* a formula that we now call **Heron's formula**:

If A is the area of the triangle with sides of length a , b , and c , and the *semiperimeter*, s , is one-half the perimeter, that is, $s = \frac{1}{2}(a + b + c)$, then

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

In *Metrica*, Heron also provided a method of finding the approximate value of the square root of a number. This method, often called the *divide and average method*, continued to be used until calculators made the pencil and paper computation of a square root unnecessary.

5-1 LINE SEGMENTS ASSOCIATED WITH TRIANGLES

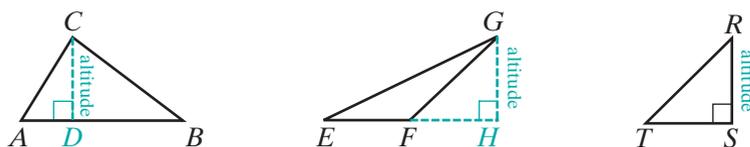
Natalie is planting a small tree. Before filling in the soil around the tree, she places stakes on opposite sides of the tree at equal distances from the base of the tree. Then she fastens cords from the same point on the trunk of the tree to the stakes. The cords are not of equal length. Natalie reasons that the tree is not perpendicular to the ground and straightens the tree until the cords are of equal lengths. Natalie used her knowledge of geometry to help her plant the tree. What was the reasoning that helped Natalie to plant the tree?

Geometric shapes are all around us. Frequently we use our knowledge of geometry to make decisions in our daily life. In this chapter you will write formal and informal proofs that will enable you to form the habit of looking for logical relationships before making a decision.

Altitude of a Triangle

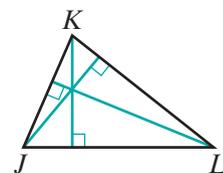
DEFINITION

An **altitude of a triangle** is a line segment drawn from any vertex of the triangle, perpendicular to and ending in the line that contains the opposite side.



In $\triangle ABC$, if \overline{CD} is perpendicular to \overline{AB} , then \overline{CD} is the altitude from vertex C to the opposite side. In $\triangle EFG$, if \overline{GH} is perpendicular to \overleftrightarrow{EF} , the line that contains the side \overline{EF} , then \overline{GH} is the altitude from vertex G to the opposite side. In an obtuse triangle such as $\triangle EFG$ above, the altitude from each of the acute angles lies outside the triangle.

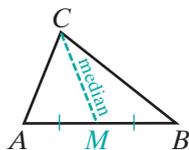
In right $\triangle TSR$, if \overline{RS} is perpendicular to \overline{TS} , then \overline{RS} is the altitude from vertex R to the opposite side \overline{TS} and \overline{TS} is the altitude from T to the opposite side \overline{RS} . In a right triangle such as $\triangle TSR$ above, the altitude from each vertex of an acute angle is a leg of the triangle. Every triangle has three altitudes as shown in $\triangle JKL$.



Median of a Triangle

DEFINITION

A **median of a triangle** is a line segment that joins any vertex of the triangle to the midpoint of the opposite side.

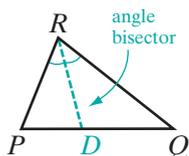


In $\triangle ABC$, if M is the midpoint of \overline{AB} , then \overline{CM} is the median drawn from vertex C to side \overline{AB} . We may also draw a median from vertex A to the midpoint of side \overline{BC} , and a median from vertex B to the midpoint of side \overline{AC} . Thus, every triangle has three medians.

Angle Bisector of a Triangle

DEFINITION

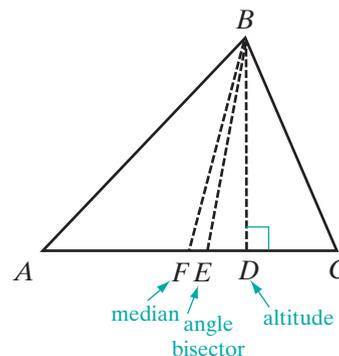
An **angle bisector of a triangle** is a line segment that bisects any angle of the triangle and terminates in the side opposite that angle.



In $\triangle PQR$, if D is a point on \overline{PQ} such that $\angle PRD \cong \angle QRD$, then \overline{RD} is the angle bisector from R in $\triangle PQR$. We may also draw an angle bisector from the vertex P to some point on \overline{QR} , and an angle bisector from the vertex Q to some point on \overline{PR} . Thus, every triangle has three angle bisectors.

In a scalene triangle, the altitude, the median, and the angle bisector drawn from any common vertex are three distinct line segments. In $\triangle ABC$, from the common vertex B , three line segments are drawn:

- \overline{BD} is the altitude from B because $\overline{BD} \perp \overline{AC}$.
- \overline{BE} is the angle bisector from B because $\angle ABE \cong \angle EBC$.
- \overline{BF} is the median from B because F is the midpoint of \overline{AC} .

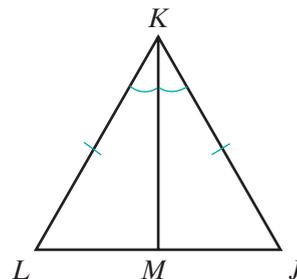


In some special triangles, such as an isosceles triangle and an equilateral triangle, some of these segments coincide, that is, are the same line. We will consider these examples later.

EXAMPLE I

Given: \overline{KM} is the angle bisector from K in $\triangle JKL$,
and $\overline{LK} \cong \overline{JK}$.

Prove: $\triangle JKM \cong \triangle LKM$



Proof	Statements	Reasons
	1. $\overline{LK} \cong \overline{JK}$	1. Given.
	2. \overline{KM} is the angle bisector from K in $\triangle JKL$.	2. Given.
	3. \overline{KM} bisects $\angle JKL$.	3. Definition of an angle bisector of a triangle.
	4. $\angle JKM \cong \angle LKM$	4. Definition of the bisector of an angle.
	5. $\overline{KM} \cong \overline{KM}$	5. Reflexive property of congruence.
	6. $\triangle JKM \cong \triangle LKM$	6. SAS (steps 1, 4, 5). ■

Exercises

Writing About Mathematics

1. Explain why the three altitudes of a right triangle intersect at the vertex of the right angle.
2. Triangle ABC is a triangle with $\angle C$ an obtuse angle. Where do the lines containing the three altitudes of the triangle intersect?

Developing Skills

3. Use a pencil, ruler, and protractor, or use geometry software, to draw $\triangle ABC$, an acute, scalene triangle with altitude \overline{CD} , angle bisector \overline{CE} , and median \overline{CF} .



- a. Name two congruent angles that have their vertices at C .
- b. Name two congruent line segments.
- c. Name two perpendicular line segments.
- d. Name two right angles.

4. Use a pencil, ruler, and protractor, or use geometry software, to draw several triangles. Include acute, obtuse, and right triangles.



- a. Draw three altitudes for each triangle.
- b. Make a conjecture regarding the intersection of the lines containing the three altitudes.

5. Use a pencil, ruler, and protractor, or use geometry software, to draw several triangles. Include acute, obtuse, and right triangles.



- a. Draw three angle bisectors for each triangle.
- b. Make a conjecture regarding the intersection of these three angle bisectors.

6. Use a pencil, ruler, and protractor, or use geometry software, to draw several triangles. Include acute, obtuse, and right triangles.



- a. Draw three medians to each triangle.
b. Make a conjecture regarding the intersection of these three medians.

In 7–9, draw and label each triangle described. Complete each required proof in two-column format.

7. *Given:* In $\triangle PQR$, $\overline{PR} \cong \overline{QR}$, $\angle P \cong \angle Q$,
and \overline{RS} is a median.

Prove: $\triangle PSR \cong \triangle QSR$

8. *Given:* In $\triangle DEF$, \overline{EG} is both an angle bisector and an altitude.

Prove: $\triangle DEG \cong \triangle FEG$

9. *Given:* \overline{CD} is a median of $\triangle ABC$ but $\triangle ADC$ is not congruent to $\triangle BDC$.

Prove: \overline{CD} is not an altitude of $\triangle ABC$.

(*Hint:* Use an indirect proof.)

Applying Skills

In 10–13, complete each required proof in paragraph format.

10. In a scalene triangle, $\triangle LNM$, show that an altitude, \overline{NO} , cannot be an angle bisector.
(*Hint:* Use an indirect proof.)
11. A telephone pole is braced by two wires that are fastened to the pole at point C and to the ground at points A and B . The base of the pole is at point D , the midpoint of \overline{AB} . If the pole is perpendicular to the ground, are the wires of equal length? Justify your answer.
12. The formula for the area of a triangle is $A = \frac{1}{2}bh$ with b the length of one side of a triangle and h the length of the altitude to that side. In $\triangle ABC$, \overline{CD} is the altitude from vertex C to side \overline{AB} and M is the midpoint of \overline{AB} . Show that the median separates $\triangle ABC$ into two triangles of equal area, $\triangle AMC$ and $\triangle BMC$.
13. A farmer has a triangular piece of land that he wants to separate into two sections of equal area. How can the land be divided?

5-2 USING CONGRUENT TRIANGLES TO PROVE LINE SEGMENTS CONGRUENT AND ANGLES CONGRUENT

The definition of congruent triangles tells us that when two triangles are congruent, each pair of corresponding sides are congruent and each pair of corresponding angles are congruent. We use three pairs of corresponding parts, SAS, ASA, or SSS, to prove that two triangles are congruent. We can then conclude that each of the other three pairs of corresponding parts are also congruent. In this section we will prove triangles congruent in order to prove that two line segments or two angles are congruent.

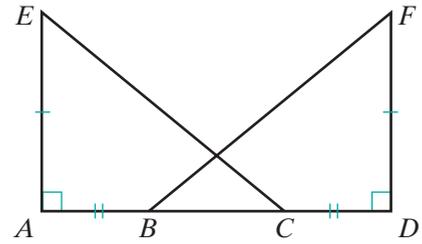
EXAMPLE I

Given: \overline{ABCD} , $\angle A$ is a right angle,
 $\angle D$ is a right angle, $\overline{AE} \cong \overline{DF}$,
 $\overline{AB} \cong \overline{CD}$.

Prove: $\overline{EC} \cong \overline{FB}$

Proof The line segments that we want to prove congruent are corresponding sides of $\triangle EAC$ and $\triangle FDB$.

Therefore we will first prove that $\triangle EAC \cong \triangle FDB$. Then use that corresponding parts of congruent triangles are congruent.



Statements	Reasons
1. $\angle A$ is a right angle, $\angle D$ is a right angle.	1. Given.
2. $\angle A \cong \angle D$	2. If two angles are right angles, then they are congruent. (Theorem 4.1)
3. $\overline{AE} \cong \overline{DF}$	3. Given.
4. $\overline{AB} \cong \overline{CD}$	4. Given.
5. $\overline{AB} + \overline{BC} \cong \overline{BC} + \overline{CD}$	5. Addition postulate.
6. \overline{ABCD}	6. Given.
7. $\overline{AB} + \overline{BC} = \overline{AC}$ $\overline{BC} + \overline{CD} = \overline{BD}$	7. Partition postulate.
8. $\overline{AC} \cong \overline{BD}$	8. Substitution postulate (steps 5, 7).
9. $\triangle EAC \cong \triangle FDB$	9. SAS (steps 3, 2, 8).
10. $\overline{EC} \cong \overline{FB}$	10. Corresponding parts of congruent triangles are congruent. ■

Exercises

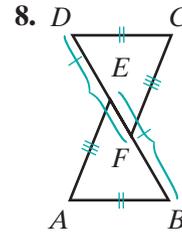
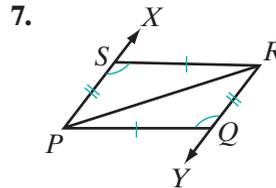
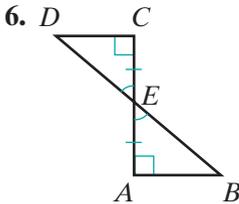
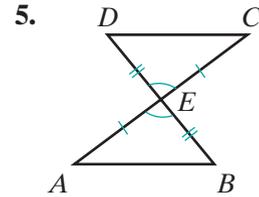
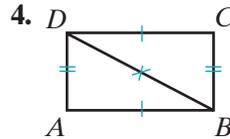
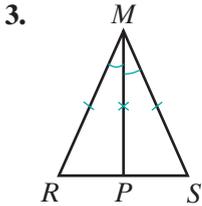
Writing About Mathematics

- Triangles ABC and DEF are congruent triangles. If $\angle A$ and $\angle B$ are complementary angles, are $\angle D$ and $\angle E$ also complementary angles? Justify your answer.
- A leg and the vertex angle of one isosceles triangle are congruent respectively to a leg and the vertex angle of another isosceles triangle. Is this sufficient information to conclude that the triangles must be congruent? Justify your answer.

Developing Skills

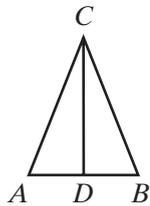
In 3–8, the figures have been marked to indicate pairs of congruent angles and pairs of congruent segments.

- In each figure, name two triangles that are congruent.
- State the reason why the triangles are congruent.
- For each pair of triangles, name three additional pairs of parts that are congruent because they are corresponding parts of congruent triangles.



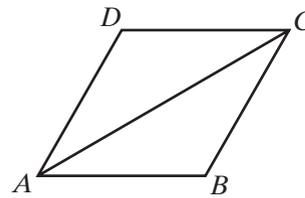
9. *Given:* $\overline{CA} \cong \overline{CB}$ and D is the midpoint of \overline{AB} .

Prove: $\angle A \cong \angle B$



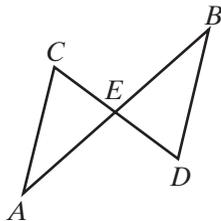
10. *Given:* $\overline{AB} \cong \overline{CD}$ and $\angle CAB \cong \angle ACD$

Prove: $\overline{AD} \cong \overline{CB}$



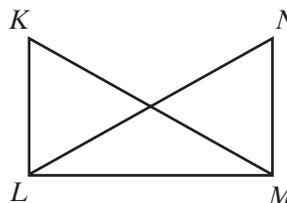
11. *Given:* \overline{AEB} and \overline{CED} bisect each other.

Prove: $\angle C \cong \angle D$



12. *Given:* $\angle KLM$ and $\angle NML$ are right angles and $KL = NM$.

Prove: $\angle K \cong \angle N$



13. Triangle ABC is congruent to triangle DEF , $AB = 3x + 7$, $DE = 5x - 9$, and $BC = 4x$. Find:
- a. x b. AB c. BC d. EF
14. Triangle PQR is congruent to triangle LMN , $m\angle P = 7a$, $m\angle L = 4a + 15$, and $\angle P$ and $\angle Q$ are complementary. Find:
- a. a b. $m\angle P$ c. $m\angle Q$ d. $m\angle M$

Applying Skills

In 15 and 16, complete each required proof in paragraph format.

15. a. Prove that the median from the vertex angle of an isosceles triangle separates the triangle into two congruent triangles.
 b. Prove that the two congruent triangles in **a** are right triangles.
16. a. Prove that if each pair of opposite sides of a quadrilateral are congruent, then a diagonal of the quadrilateral separates it into two congruent triangles.
 b. Prove that a pair of opposite angles of the quadrilateral in **a** are congruent.

In 17 and 18, complete each required proof in two-column format.

17. a. Points B and C separate \overline{ABCD} into three congruent segments. P is a point not on \overleftrightarrow{AD} such that $\overline{PA} \cong \overline{PD}$ and $\overline{PB} \cong \overline{PC}$. Draw a diagram that shows these line segments and write the information in a *given* statement.
 b. Prove that $\angle APB \cong \angle DPC$.
 c. Prove that $\angle APC \cong \angle DPB$.
18. The line segment \overline{PM} is both the altitude and the median from P to \overline{LN} in $\triangle LNP$.
 a. Prove that $\triangle LNP$ is isosceles.
 b. Prove that \overline{PM} is also the angle bisector from P in $\triangle LNP$.

5-3 ISOSCELES AND EQUILATERAL TRIANGLES

When working with triangles, we observed that when two sides of a triangle are congruent, the median, the altitude, and the bisector of the vertex angle separate the triangle into two congruent triangles. These congruent triangles can be used to prove that the base angles of an isosceles triangle are congruent. This observation can be proved as a theorem called the **Isosceles Triangle Theorem**.

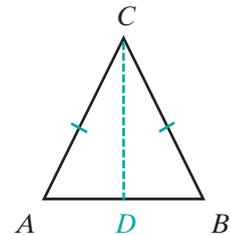
Theorem 5.1

If two sides of a triangle are congruent, the angles opposite these sides are congruent.

Given $\triangle ABC$ with $\overline{AC} \cong \overline{BC}$

Prove $\angle A \cong \angle B$

Proof In order to prove this theorem, we will use the median to the base to separate the triangle into two congruent triangles.



Statements	Reasons
1. Draw D , the midpoint of \overline{AB} .	1. A line segment has one and only one midpoint.
2. \overline{CD} is the median from vertex C .	2. Definition of a median of a triangle.
3. $\overline{CD} \cong \overline{CD}$	3. Reflexive property of congruence.
4. $\overline{AD} \cong \overline{DB}$	4. Definition of a midpoint.
5. $\overline{AC} \cong \overline{BC}$	5. Given.
6. $\triangle ACD \cong \triangle BCD$	6. SSS (steps 3, 4, 5).
7. $\angle A \cong \angle B$	7. Corresponding parts of congruent triangles are congruent. ■

A **corollary** is a theorem that can easily be deduced from another theorem. We can prove two other statements that are corollaries of the isosceles triangle theorem because their proofs follow directly from the proof of the theorem.

Corollary 5.1a

The median from the vertex angle of an isosceles triangle bisects the vertex angle.

Proof: From the preceding proof that $\triangle ACD \cong \triangle BCD$, we can also conclude that $\angle ACD \cong \angle BCD$ since they, too, are corresponding parts of congruent triangles. ■

Corollary 5.1b

The median from the vertex angle of an isosceles triangle is perpendicular to the base.

Proof: Again, from $\triangle ACD \cong \triangle BCD$, we can say that $\angle CDA \cong \angle CDB$ because they are corresponding parts of congruent triangles. If two lines intersect to form congruent adjacent angles, then they are perpendicular. Therefore, $\overline{CD} \perp \overline{AB}$. ■

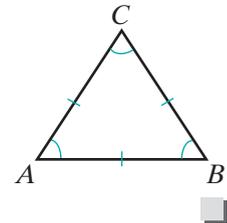
Properties of an Equilateral Triangle

The isosceles triangle theorem has shown that in an isosceles triangle with two congruent sides, the angles opposite these sides are congruent. We may prove another corollary to this theorem for any equilateral triangle, where three sides are congruent.

Corollary 5.1c

Every equilateral triangle is equiangular.

Proof: If $\triangle ABC$ is equilateral, then $\overline{AB} \cong \overline{BC} \cong \overline{CA}$. By the isosceles triangle theorem, since $\overline{AB} \cong \overline{BC}$, $\angle A \cong \angle C$, and since $\overline{BC} \cong \overline{CA}$, $\angle B \cong \angle A$. Therefore, $\angle A \cong \angle B \cong \angle C$.

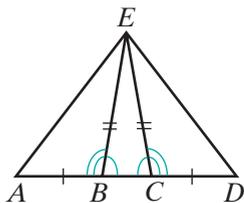
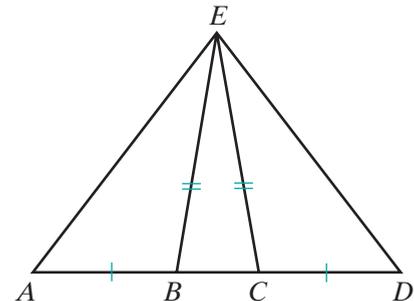


EXAMPLE 1

Given: E not on \overline{ABCD} , $\overline{AB} \cong \overline{CD}$, and $\overline{EB} \cong \overline{EC}$.

Prove: $\overline{AE} \cong \overline{DE}$

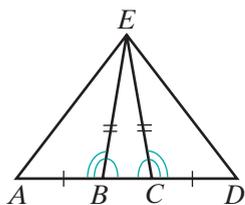
Proof \overline{AE} and \overline{DE} are corresponding sides of $\triangle ABE$ and $\triangle DCE$, and we will prove these triangles congruent by SAS. We are given two pairs of congruent corresponding sides and must prove that the included angles are congruent.



Statements	Reasons
1. $\overline{EB} \cong \overline{EC}$	1. Given.
2. $\angle EBC \cong \angle ECB$	2. Isosceles triangle theorem (Or: If two sides of a triangle are congruent, the angles opposite these sides are congruent.).
3. $\overline{AB} \cong \overline{CD}$	3. Given.

Continued

(Continued)



Statements	Reasons
4. $\angle ABE$ and $\angle EBC$ are supplementary. $\angle DCE$ and $\angle ECB$ are supplementary.	4. If two angles form a linear pair, they are supplementary.
5. $\angle ABE \cong \angle DCE$	5. The supplements of congruent angles are congruent.
6. $\overline{AB} \cong \overline{CD}$	6. Given.
7. $\triangle ABE \cong \triangle DCE$	7. SAS (steps 1, 5, 6).
8. $\overline{AE} \cong \overline{DE}$	8. Corresponding parts of congruent triangles are congruent. ■

Exercises

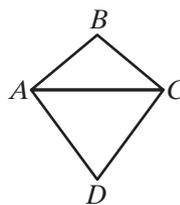
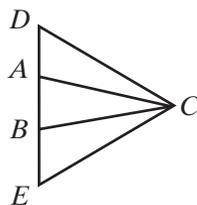
Writing About Mathematics

- Joel said that the proof given in Example 1 could have been done by proving that $\triangle ACE \cong \triangle DBE$. Do you agree with Joel? Justify your answer.
- Abel said that he could prove that equiangular triangle ABC is equilateral by drawing median \overline{BD} and showing that $\triangle ABD \cong \triangle CBD$. What is wrong with Abel's reasoning?

Developing Skills

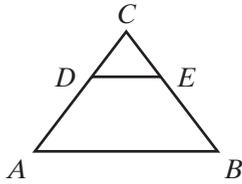
- In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$, $m\angle B = 3x + 15$ and $m\angle C = 7x - 5$, find $m\angle B$ and $m\angle C$.
- Triangle RST is an isosceles right triangle with $RS = ST$ and $\angle R$ and $\angle T$ complementary angles. What is the measure of each angle of the triangle?
- In equilateral $\triangle DEF$, $m\angle D = 3x + y$, $m\angle E = 2x + 40$, and $m\angle F = 2y$. Find x , y , $m\angle D$, $m\angle E$, and $m\angle F$.
- Given: C not on \overline{DABE} and $\overline{CA} \cong \overline{CB}$
- Given: Quadrilateral $ABCD$ with $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$

 Prove: $\angle CAD \cong \angle CBE$

 Prove: $\angle BAD \cong \angle BCD$


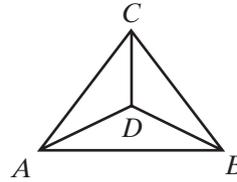
8. Given: $\overline{AC} \cong \overline{CB}$ and $\overline{DA} \cong \overline{EB}$

Prove: $\angle CDE \cong \angle CED$



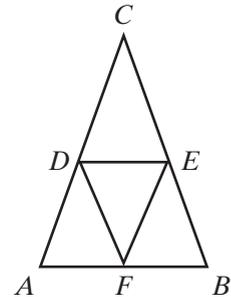
9. Given: $\overline{AC} \cong \overline{BC}$ and $\angle DAB \cong \angle DBA$

Prove: $\angle CAD \cong \angle CBD$



10. Given: Isosceles $\triangle ABC$ with $\overline{AC} \cong \overline{BC}$, D is the midpoint of \overline{AC} , E is the midpoint of \overline{BC} and F is the midpoint of \overline{AB} .

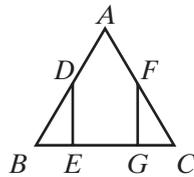
- a. Prove: $\triangle ADF \cong \triangle BEF$
 b. Prove: $\triangle DEF$ is isosceles.



In 11 and 12, complete each given proof with a partner or in a small group.

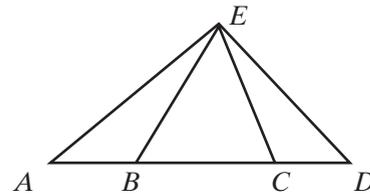
11. Given: $\triangle ABC$ with $AB = AC$, $BG = EC$,
 $\overline{BE} \perp \overline{DE}$, and $\overline{CG} \perp \overline{GF}$.

Prove: $\overline{BD} \cong \overline{CF}$



12. Given: E not on \overline{ABCD} , $\overline{AB} \cong \overline{CD}$,
 and \overline{EB} is not congruent to \overline{EC} .

Prove: \overline{AE} is not congruent to \overline{DE} .



(Hint: Use an indirect proof.)

Applying Skills

13. Prove the isosceles triangle theorem by drawing the bisector of the vertex angle instead of the median.
14. Prove that the line segments joining the midpoints of the sides of an equilateral triangle are congruent.
15. C is a point not on $\overleftrightarrow{FBDG}$ and $BC = DC$. Prove that $\angle FBC \cong \angle GDC$.
16. In $\triangle PQR$, $m\angle R \neq m\angle Q$. Prove that $PQ \neq PR$.

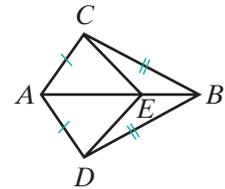
5-4 USING TWO PAIRS OF CONGRUENT TRIANGLES

Often the line segments or angles that we want to prove congruent are not corresponding parts of triangles that can be proved congruent by using the given information. However, it may be possible to use the given information to prove a different pair of triangles congruent. Then the congruent corresponding parts of this second pair of triangles can be used to prove the required triangles congruent. The following is an example of this method of proof.

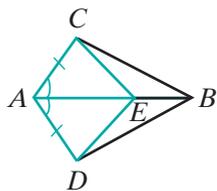
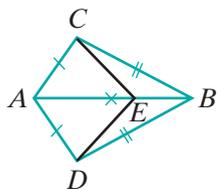
EXAMPLE 1

Given: $\overline{AC} \cong \overline{AD}$, and $\overline{CB} \cong \overline{DB}$

Prove: $\overline{CE} \cong \overline{DE}$



Proof Since \overline{CE} and \overline{DE} are corresponding parts of $\triangle ACE$ and $\triangle ADE$, we can prove these two line segments congruent if we can prove $\triangle ACE$ and $\triangle ADE$ congruent. From the given, we cannot prove immediately that $\triangle ACE$ and $\triangle ADE$ congruent. However, we can prove that $\triangle CAB \cong \triangle DAB$. Using corresponding parts of these larger congruent triangles, we can then prove that the smaller triangles are congruent.



Statements	Reasons
1. $\overline{AC} \cong \overline{AD}$	1. Given.
2. $\overline{CB} \cong \overline{DB}$	2. Given.
3. $\overline{AB} \cong \overline{AB}$	3. Reflexive property of congruence.
4. $\triangle CAB \cong \triangle DAB$	4. SSS (steps 1, 2, 3).
5. $\angle CAB \cong \angle DAB$	5. Corresponding parts of congruent triangles are congruent.
6. $\overline{AE} \cong \overline{AE}$	6. Reflexive property of congruence.
7. $\triangle CAE \cong \triangle DAE$	7. SAS (steps 1, 5, 6).
8. $\overline{CE} \cong \overline{DE}$	8. Corresponding parts of congruent triangles are congruent.

Exercises

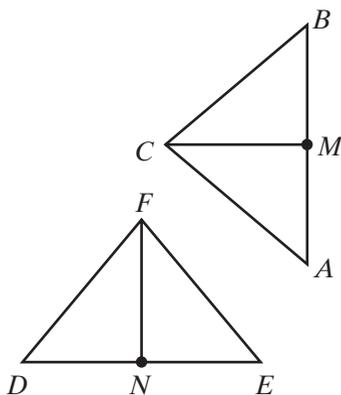
Writing About Mathematics

1. Can Example 1 be proved by proving that $\triangle BCE \cong \triangle BDE$? Justify your answer.
2. Greg said that if it can be proved that two triangles are congruent, then it can be proved that the medians to corresponding sides of these triangles are congruent. Do you agree with Greg? Explain why or why not.

Developing Skills

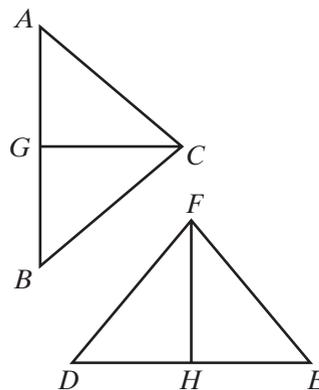
3. *Given:* $\triangle ABC \cong \triangle DEF$, M is the midpoint of \overline{AB} , and N is the midpoint of \overline{DE} .

Prove: $\triangle AMC \cong \triangle DNF$



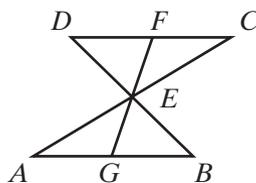
4. *Given:* $\triangle ABC \cong \triangle DEF$, \overline{CG} bisects $\angle ACB$, and \overline{FH} bisects $\angle DFE$.

Prove: $\overline{CG} \cong \overline{FH}$



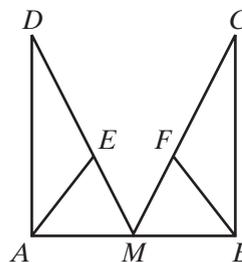
5. *Given:* \overline{AEC} and \overline{DEB} bisect each other, \overline{FEG} intersects \overline{AB} at G and \overline{CD} at F .

Prove: E is the midpoint of \overline{FG} .



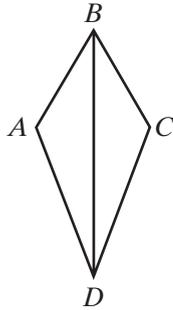
6. *Given:* $\triangle AME \cong \triangle BMF$ and $\overline{DE} \cong \overline{CF}$

Prove: $\overline{AD} \cong \overline{BC}$



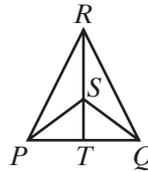
7. Given: $\overline{BC} \cong \overline{BA}$ and \overrightarrow{BD} bisects $\angle CBA$.

Prove: \overrightarrow{DB} bisects $\angle CDA$.



8. Given: $RP = RQ$ and $SP = SQ$

Prove: $\overline{RT} \perp \overline{PQ}$



Applying Skills

9. In quadrilateral $ABCD$, $AB = CD$, $BC = DA$, and M is the midpoint of \overline{BD} . A line segment through M intersects \overline{AB} at E and \overline{CD} at F . Prove that \overline{BMD} bisects \overline{EMF} at M .

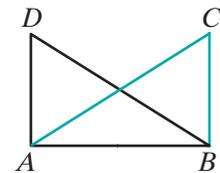
10. Complete the following exercise with a partner or in a small group:

Line l intersects \overline{AB} at M , and P and S are any two points on l . Prove that if $PA = PB$ and $SA = SB$, then M is the midpoint of \overline{AB} and l is perpendicular to \overline{AB} .

- Let half the group treat the case in which P and S are on the same side of \overline{AB} .
- Let half the group treat the case in which P and S are on opposite sides of \overline{AB} .
- Compare and contrast the methods used to prove the cases.

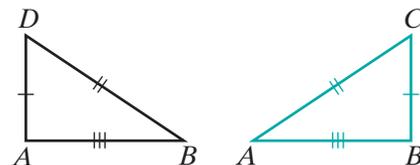
5-5 PROVING OVERLAPPING TRIANGLES CONGRUENT

If we know that $\overline{AD} \cong \overline{BC}$ and $\overline{DB} \cong \overline{CA}$, can we prove that $\triangle DBA \cong \triangle CAB$? These two triangles overlap and share a common side. To make it easier to visualize the overlapping triangles that we want to prove congruent, it may be helpful to outline each with a different color as shown in the figure.



Or the triangles can be redrawn as separate triangles.

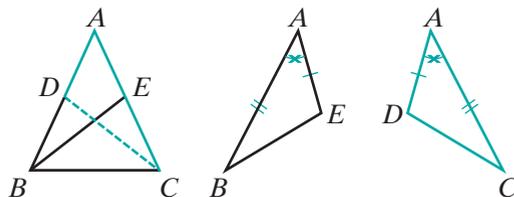
The segment \overline{AB} is a side of each of the triangles $\triangle DBA$ and $\triangle CAB$. Therefore, to the given information, $\overline{AD} \cong \overline{BC}$ and $\overline{DB} \cong \overline{CA}$, we can add $\overline{AB} \cong \overline{AB}$ and prove that $\triangle DBA \cong \triangle CAB$ by SSS.



EXAMPLE 1

Given: \overline{CD} and \overline{BE} are medians to the legs of isosceles $\triangle ABC$.

Prove: $\overline{CD} \cong \overline{BE}$



Proof Separate the triangles to see more clearly the triangles to be proved congruent. We know that the legs of an isosceles triangle are congruent. Therefore, $\overline{AB} \cong \overline{AC}$. We also know that the median is a line segment from a vertex to the midpoint of the opposite side. Therefore, D and E are midpoints of the congruent legs. The midpoint divides the line segment into two congruent segments, that is, in half, and halves of congruent segments are congruent: $\overline{AE} \cong \overline{AD}$. Now we have two pair of congruent sides: $\overline{AB} \cong \overline{AC}$ and $\overline{AE} \cong \overline{AD}$. The included angle between each of these pairs of congruent sides is $\angle A$, and $\angle A$ is congruent to itself by the reflexive property of congruence. Therefore, $\triangle ABE \cong \triangle ACD$ by SAS and $\overline{CD} \cong \overline{BE}$ because corresponding parts of congruent triangles are congruent. \blacksquare

EXAMPLE 2

Using the results of Example 1, find the length of \overline{BE} if $BE = 5x - 9$ and $CD = x + 15$.

Solution

$$\begin{aligned}
 BE &= CD \\
 5x - 9 &= x + 15 \\
 4x &= 24 \\
 x &= 6 \\
 BE &= 5x - 9 & CD &= x + 15 \\
 &= 5(6) - 9 & &= 6 + 15 \\
 &= 30 - 9 & &= 21 \\
 &= 21 & &
 \end{aligned}$$

Answer 21 \blacksquare

Exercises
Writing About Mathematics

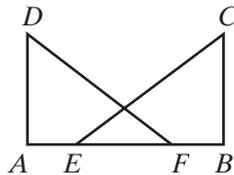
- In Example 1, the medians to the legs of isosceles $\triangle ABC$ were proved to be congruent by proving $\triangle ABE \cong \triangle ACD$. Could the proof have been done by proving $\triangle DBC \cong \triangle ECB$? Justify your answer.

2. In Corollary 5.1b, we proved that the median to the base of an isosceles triangle is also the altitude to the base. If the median to a leg of an isosceles triangle is also the altitude to the leg of the triangle, what other type of triangle must this triangle be?

Developing Skills

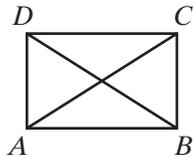
3. Given: \overline{AEFB} , $\overline{AE} \cong \overline{FB}$, $\overline{DA} \cong \overline{CB}$,
and $\angle A$ and $\angle B$ are right angles.

Prove: $\triangle DAF \cong \triangle CBE$ and $\overline{DF} \cong \overline{CE}$



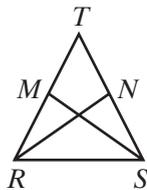
5. Given: $\overline{DA} \cong \overline{CB}$, $\overline{DA} \perp \overline{AB}$, and
 $\overline{CB} \perp \overline{AB}$

Prove: $\triangle DAB \cong \triangle CBA$ and $\overline{AC} \cong \overline{BD}$



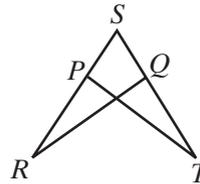
7. Given: $\overline{TM} \cong \overline{TN}$, M is the midpoint of \overline{TR}
and N is the midpoint of \overline{TS} .

Prove: $\overline{RN} \cong \overline{SM}$



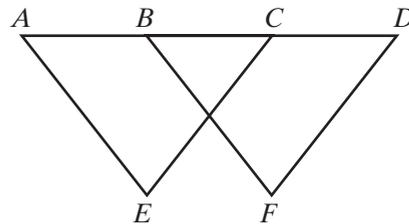
4. Given: $\overline{SPR} \cong \overline{SQT}$, $\overline{PR} \cong \overline{QT}$

Prove: $\triangle SRQ \cong \triangle STP$ and $\angle R \cong \angle T$



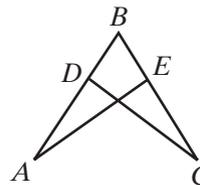
6. Given: \overline{ABCD} , $\angle BAE \cong \angle CBF$,
 $\angle BCE \cong \angle CDF$, $\overline{AB} \cong \overline{CD}$

Prove: $\overline{AE} \cong \overline{BF}$ and $\angle E \cong \angle F$



8. Given: $\overline{AD} \cong \overline{CE}$ and $\overline{DB} \cong \overline{EB}$

Prove: $\angle ADC \cong \angle CEA$



Applying Skills

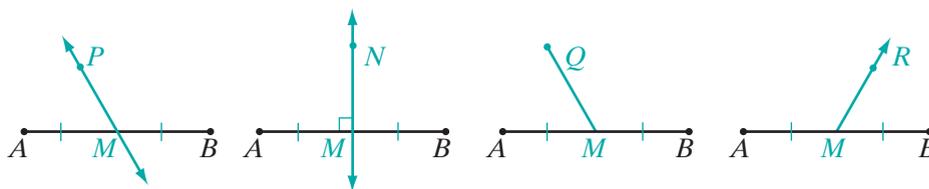
In 9–11, complete each required proof in paragraph format.

9. Prove that the angle bisectors of the base angles of an isosceles triangle are congruent.
10. Prove that the triangle whose vertices are the midpoints of the sides of an isosceles triangle is an isosceles triangle.
11. Prove that the median to any side of a scalene triangle is *not* the altitude to that side.

5-6 PERPENDICULAR BISECTOR OF A LINE SEGMENT

Perpendicular lines were defined as lines that intersect to form right angles. We also proved that if two lines intersect to form congruent adjacent angles, then they are perpendicular. (Theorem 4.8)

The bisector of a line segment was defined as any line or subset of a line that intersects a line segment at its midpoint.



In the diagrams, \overleftrightarrow{PM} , \overleftrightarrow{NM} , \overleftrightarrow{QM} , and \overleftrightarrow{MR} are all bisectors of \overline{AB} since they each intersect \overline{AB} at its midpoint, M . Only \overleftrightarrow{NM} is both perpendicular to \overline{AB} and the bisector of \overline{AB} . \overleftrightarrow{NM} is the *perpendicular bisector* of \overline{AB} .

DEFINITION

The **perpendicular bisector of a line segment** is any line or subset of a line that is perpendicular to the line segment at its midpoint.

In Section 3 of this chapter, we proved as a corollary to the isosceles triangle theorem that the median from the vertex angle of an isosceles triangle is perpendicular to the base.

In the diagram below, since \overline{CM} is the median to the base of isosceles $\triangle ABC$, $\overline{CM} \perp \overline{AB}$. Therefore, \overleftrightarrow{CM} is the perpendicular bisector of \overline{AB} .

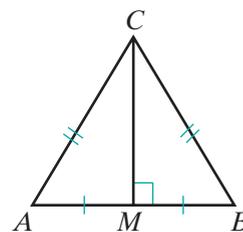
(1) M is the midpoint of \overline{AB} : $AM = MB$.

M is **equidistant**, or is at an equal distance, from the endpoints of \overline{AB} .

(2) \overline{AB} is the base of isosceles $\triangle ABC$: $AC = BC$.

C is equidistant from the endpoints of \overline{AB} .

These two points, M and C , determine the perpendicular bisector of \overline{AB} . This suggests the following theorem.

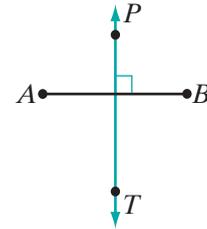


Theorem 5.2

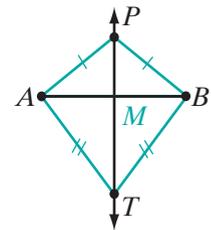
If two points are each equidistant from the endpoints of a line segment, then the points determine the perpendicular bisector of the line segment.

Given \overline{AB} and points P and T such that $PA = PB$ and $TA = TB$.

Prove \overleftrightarrow{PT} is the perpendicular bisector of \overline{AB} .



Strategy Let \overleftrightarrow{PT} intersect \overline{AB} at M . Prove $\triangle APT \cong \triangle BPT$ by SSS. Then, using the congruent corresponding angles, prove $\triangle APM \cong \triangle BPM$ by SAS. Consequently, $\overline{AM} \cong \overline{MB}$, so \overleftrightarrow{PT} is a bisector. Also, $\angle AMP \cong \angle BMP$. Since two lines that intersect to form congruent adjacent angles are perpendicular, $\overline{AB} \perp \overleftrightarrow{PT}$. Therefore, \overleftrightarrow{PT} is the perpendicular bisector of \overline{AB} .



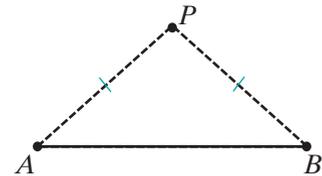
The details of the proof of Theorem 5.2 will be left to the student. (See exercise 7.)

Theorem 5.3a

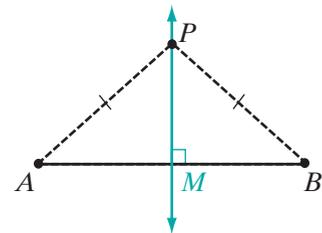
If a point is equidistant from the endpoints of a line segment, then it is on the perpendicular bisector of the line segment.

Given Point P such that $PA = PB$.

Prove P lies on the perpendicular bisector of \overline{AB} .



Proof Choose any other point that is equidistant from the endpoints of \overline{AB} , for example, M , the midpoint of \overline{AB} . Then \overleftrightarrow{PM} is the perpendicular bisector of \overline{AB} by Theorem 5.2. (If two points are each equidistant from the endpoints of a line segment, then the points determine the perpendicular bisector of the line segment.) P lies on \overleftrightarrow{PM} . ■



The converse of this theorem is also true.

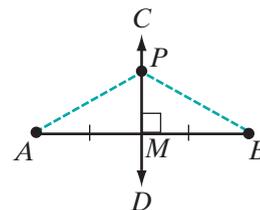
Theorem 5.3b

If a point is on the perpendicular bisector of a line segment, then it is equidistant from the endpoints of the line segment.

Given Point P on the perpendicular bisector of \overline{AB} .

Prove $PA = PB$

Proof Let M be the midpoint of \overline{AB} . Then $\overline{AM} \cong \overline{BM}$ and $\overline{PM} \cong \overline{PM}$. Perpendicular lines intersect to form right angles, so $\angle PMA \cong \angle PMB$. By SAS, $\triangle PMA \cong \triangle PMB$. Since corresponding parts of congruent triangles are congruent, $\overline{PA} \cong \overline{PB}$ and $PA = PB$. ■



Theorems 5.3a and 5.3b can be written as a biconditional.

Theorem 5.3

A point is on the perpendicular bisector of a line segment if and only if it is equidistant from the endpoints of the line segment.

Methods of Proving Lines or Line Segments Perpendicular

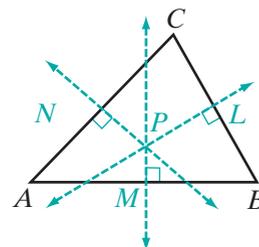
To prove that two intersecting lines or line segments are perpendicular, prove that one of the following statements is true:

1. The two lines form right angles at their point of intersection.
2. The two lines form congruent adjacent angles at their point of intersection.
3. Each of two points on one line is equidistant from the endpoints of a segment of the other.

Intersection of the Perpendicular Bisectors of the Sides of a Triangle

When we draw the three perpendicular bisectors of the sides of a triangle, it appears that the three lines are **concurrent**, that is, they intersect in one point.

Theorems 5.3a and 5.3b allow us to prove the following **perpendicular bisector concurrence theorem**.



Theorem 5.4

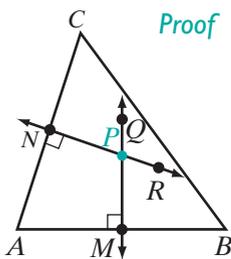
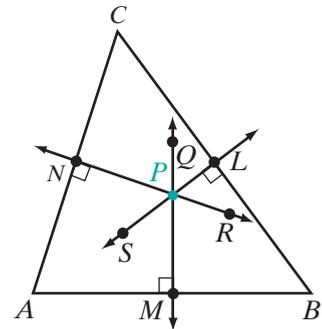
The perpendicular bisectors of the sides of a triangle are concurrent.

Given \overleftrightarrow{MQ} , the perpendicular bisector of \overline{AB}

\overleftrightarrow{NR} , the perpendicular bisector of \overline{AC}

\overleftrightarrow{LS} , the perpendicular bisector of \overline{BC}

Prove \overleftrightarrow{MQ} , \overleftrightarrow{NR} , and \overleftrightarrow{LS} intersect in P .



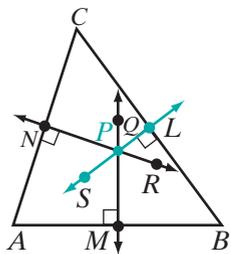
Proof (1) We can assume from the diagram that \overleftrightarrow{MQ} and \overleftrightarrow{NR} intersect. Let us call the point of intersection P .

(2) By theorem 5.3b, since P is a point on \overleftrightarrow{MQ} , the perpendicular bisector of \overline{AB} , P is equidistant from A and B .

(3) Similarly, since P is a point on \overleftrightarrow{NR} , the perpendicular bisector of \overline{AC} , P is equidistant from A and C .

(4) In other words, P is equidistant from A , B , and C . By theorem 5.3a, since P is equidistant from the endpoints of \overline{BC} , P is on the perpendicular bisector of \overline{BC} .

(5) Therefore, \overleftrightarrow{MQ} , \overleftrightarrow{NR} , and \overleftrightarrow{LS} , the three perpendicular bisectors of $\triangle ABC$, intersect in a point, P . ■



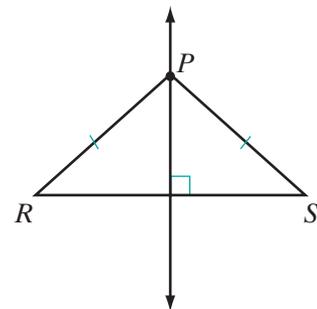
The point where the three perpendicular bisectors of the sides of a triangle intersect is called the **circumcenter**.

EXAMPLE 1

Prove that if a point lies on the perpendicular bisector of a line segment, then the point and the endpoints of the line segment are the vertices of an isosceles triangle.

Given: P lies on the perpendicular bisector of \overline{RS} .

Prove: $\triangle RPS$ is isosceles.



Proof	Statements	Reasons
	1. P lies on the perpendicular bisector of \overline{RS} .	1. Given.
	2. $PR = PS$	2. If a point is on the perpendicular bisector of a line segment, then it is equidistant from the endpoints of the line segment. (Theorem 5.3b)
	3. $\overline{PR} \cong \overline{PS}$	3. Segments that have the same measure are congruent.
	4. $\triangle RPS$ is isosceles.	4. An isosceles triangle is a triangle that has two congruent sides. ■

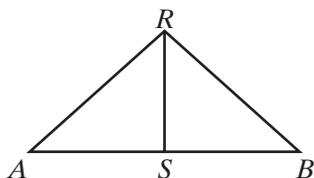
Exercises

Writing About Mathematics

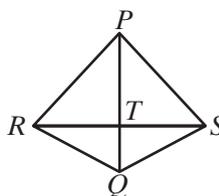
- Justify the three methods of proving that two lines are perpendicular given in this section.
- Compare and contrast Example 1 with Corollary 5.1b, “The median from the vertex angle of an isosceles triangle is perpendicular to the base.”

Developing Skills

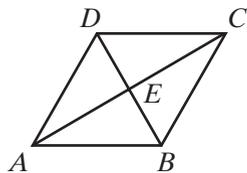
3. If \overline{RS} is the perpendicular bisector of \overline{ASB} , prove that $\angle ARS \cong \angle BRS$.



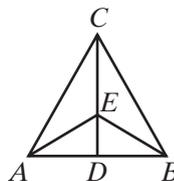
4. If $PR = PS$ and $QR = QS$, prove that $\overline{PQ} \perp \overline{RS}$ and $RT = ST$.



5. Polygon $ABCD$ is equilateral ($AB = BC = CD = DA$). Prove that \overline{AC} and \overline{BD} bisect each other and are perpendicular to each other.



6. Given \overline{CED} and \overline{ADB} with $\angle ACE \cong \angle BCE$ and $\angle AED \cong \angle BED$, prove that \overline{CED} is the perpendicular bisector of \overline{ADB} .



Applying Skills

7. Prove Theorem 5.2.
8. Prove that if the bisector of an angle of a triangle is perpendicular to the opposite side of the triangle, the triangle is isosceles.
9. A line through one vertex of a triangle intersects the opposite side of the triangle in adjacent angles whose measures are represented by $\frac{1}{2}a + 27$ and $\frac{3}{2}a - 15$. Is the line perpendicular to the side of the triangle? Justify your answer.

5-7 BASIC CONSTRUCTIONS

A **geometric construction** is a drawing of a geometric figure done using only a pencil, a *compass*, and a *straightedge*, or their equivalents. A **straightedge** is used to draw a line segment but is not used to measure distance or to determine equal distances. A **compass** is used to draw circles or arcs of circles to locate points at a fixed distance from given point.

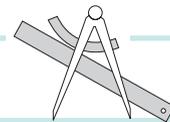
The six constructions presented in this section are the basic procedures used for all other constructions. The following postulate allows us to perform these basic constructions:

Postulate 5.1

Radii of congruent circles are congruent.

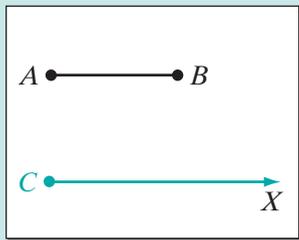
Construction 1

Construct a Line Segment Congruent to a Given Line Segment.

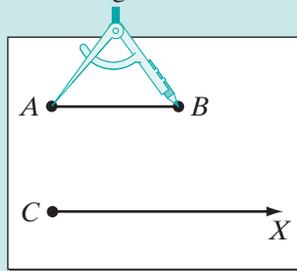


Given \overline{AB}

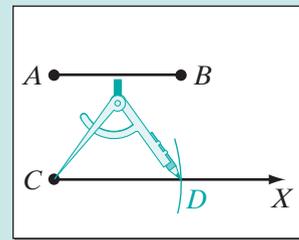
Construct \overline{CD} , a line segment congruent to \overline{AB} .



1. With a straightedge, draw a ray, \overrightarrow{CX} .



2. Open the compass so that the point is on A and the point of the pencil is on B.



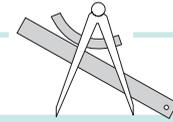
3. Using the same compass radius, place the point on C and, with the pencil, draw an arc that intersects \overrightarrow{CX} . Label this point of intersection D.

Conclusion $\overline{CD} \cong \overline{AB}$

Proof Since \overline{AB} and \overline{CD} are radii of congruent circles, they are congruent. ■

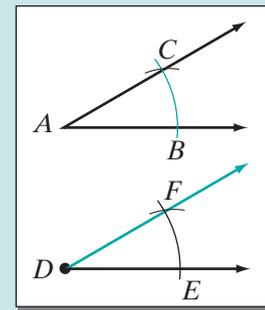
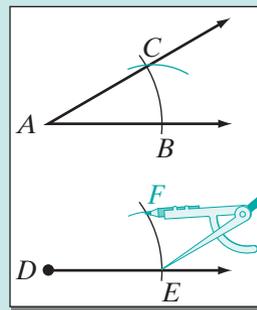
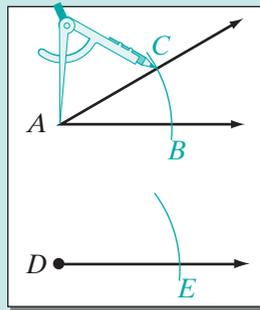
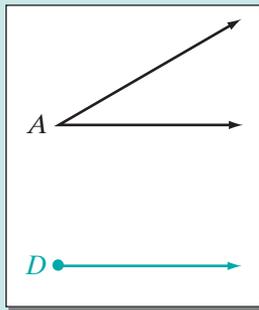
Construction 2

Construct an Angle Congruent to a Given Angle.



Given $\angle A$

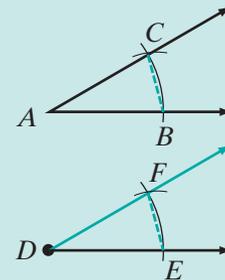
Construct $\angle EDF \cong \angle BAC$



1. Draw a ray with endpoint D .
2. With A as center, draw an arc that intersects each ray of $\angle A$. Label the points of intersection B and C . Using the same radius, draw an arc with D as the center that intersects the ray from D at E .
3. With E as the center, draw an arc with radius equal to BC that intersects the arc drawn in step 2. Label the intersection F .
4. Draw \overrightarrow{DF} .

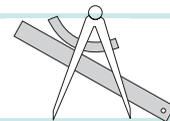
Conclusion $\angle EDF \cong \angle BAC$

Proof We used congruent radii to draw $\overline{AC} \cong \overline{DF}$, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$. Therefore, $\triangle DEF \cong \triangle ABC$ by SSS and $\angle EDF \cong \angle BAC$ because they are corresponding parts of congruent triangles.



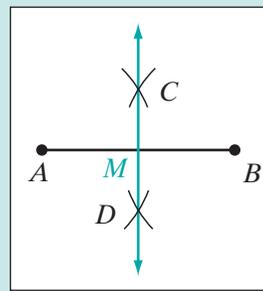
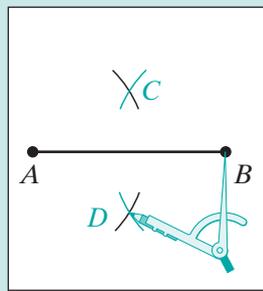
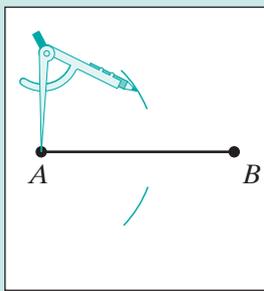
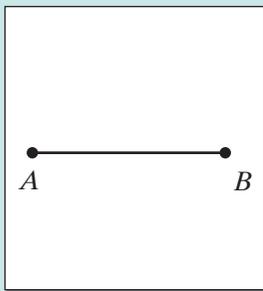
Construction 3

Construct the Perpendicular Bisector of a Given Line Segment and the Midpoint of a Given Line Segment.



Given \overline{AB}

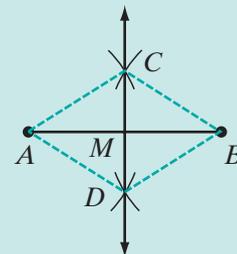
Construct $\overleftrightarrow{CD} \perp \overline{AB}$ at M , the midpoint of \overline{AB} .

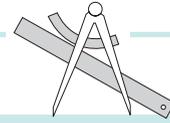


1. Open the compass to a radius that is greater than one-half of AB .
2. Place the point of the compass at A and draw an arc above \overline{AB} and an arc below \overline{AB} .
3. Using the same radius, place the point of the compass at B and draw an arc above \overline{AB} and an arc below \overline{AB} intersecting the arcs drawn in step 2. Label the intersections C and D .
4. Use a straightedge to draw \overleftrightarrow{CD} intersecting \overline{AB} at M .

Conclusion $\overleftrightarrow{CD} \perp \overline{AB}$ at M , the midpoint of \overline{AB} .

Proof Since they are congruent radii, $\overline{AC} \cong \overline{BC}$ and $\overline{AD} \cong \overline{BD}$. Therefore, C and D are both equidistant from A and B . If two points are each equidistant from the endpoints of a line segment, then the points determine the perpendicular bisector of the line segment (Theorem 5.2). Thus, \overleftrightarrow{CD} is the perpendicular bisector of \overline{AB} . Finally, M is the point on \overline{AB} where the perpendicular bisector intersects \overline{AB} , so $AM = BM$. By definition, M is the midpoint of \overline{AB} .



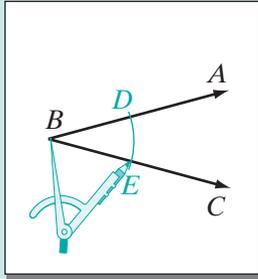


Construction 4

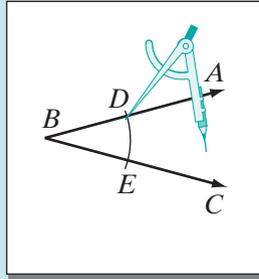
Bisect a Given Angle.

Given $\angle ABC$

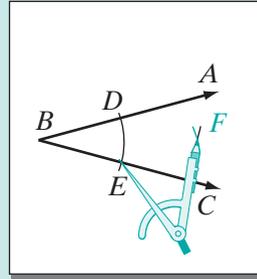
Construct \overrightarrow{BF} , the bisector of $\angle ABC$



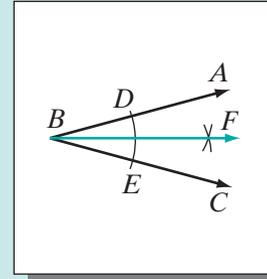
1. With B as center and any convenient radius, draw an arc that intersects \overrightarrow{BA} at D and \overrightarrow{BC} at E .



2. With D as center, draw an arc in the interior of $\angle ABC$.



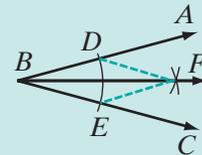
3. Using the same radius, and with E as center, draw an arc that intersects the arc drawn in step 2. Label this intersection F .



4. Draw \overrightarrow{BF} .

Conclusion \overrightarrow{BF} bisects $\angle ABC$; $\angle ABF \cong \angle FBC$.

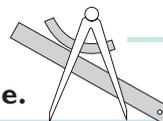
Proof We used congruent radii to draw $\overline{BD} \cong \overline{BE}$ and $\overline{DF} \cong \overline{EF}$. By the reflexive property of congruence, $\overline{BF} \cong \overline{BF}$ so by SSS, $\triangle FBD \cong \triangle FBE$. Therefore, $\angle ABF \cong \angle FBC$ because they are corresponding parts of congruent triangles. Then \overrightarrow{BF} bisects $\angle ABC$ because an angle bisector separates an angle into two congruent angles. ■



Construction 5 will be similar to Construction 4. In Construction 4, any given angle is bisected. In Construction 5, $\angle APB$ is a straight angle that is bisected by the construction. Therefore, $\angle APE$ and $\angle BPE$ are right angles and $\overrightarrow{PE} \perp \overrightarrow{AB}$.

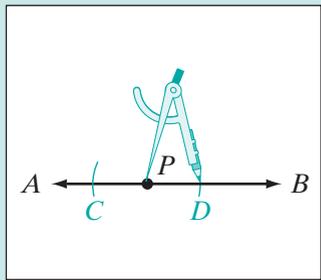
Construction 5

Construct a Line Perpendicular to a Given Line Through a Given Point on the Line.

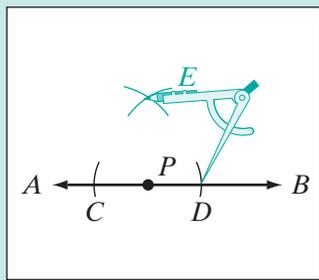


Given Point P on \overleftrightarrow{AB} .

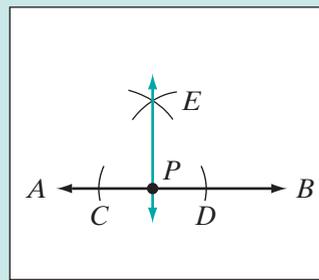
Construct $\overleftrightarrow{PE} \perp \overleftrightarrow{AB}$



1. With P as center and any convenient radius, draw arcs that intersect \overleftrightarrow{PA} at C and \overleftrightarrow{PB} at D .



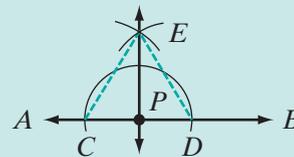
2. With C and D as centers and a radius greater than that used in step 1, draw arcs intersecting at E .

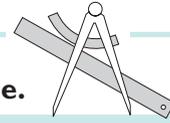


3. Draw \overleftrightarrow{EP} .

Conclusion $\overleftrightarrow{PE} \perp \overleftrightarrow{AB}$

Proof Since points C and D were constructed using congruent radii, $CP = PD$ and P is equidistant to C and D . Similarly, since E was constructed using congruent radii, $CE = ED$, and E is equidistant to C and D . If two points are each equidistant from the endpoints of a line segment, then the points determine the perpendicular bisector of the line segment (Theorem 5.2). Therefore, \overleftrightarrow{PE} is the perpendicular bisector of \overline{CD} . Since \overline{CD} is a subset of line \overleftrightarrow{AB} , $\overleftrightarrow{PE} \perp \overleftrightarrow{AB}$.



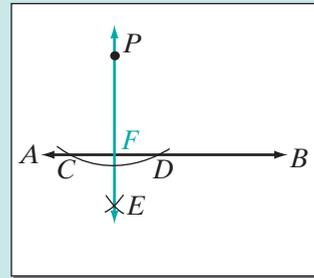
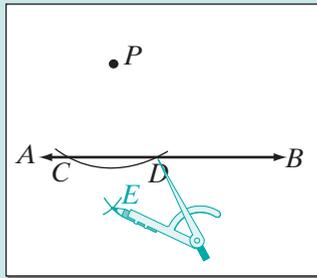
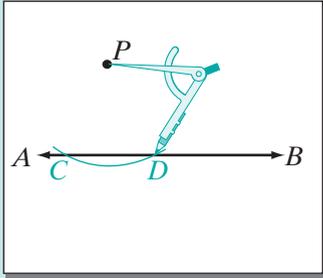


Construction 6

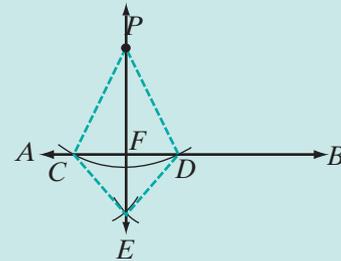
Construct a Line Perpendicular to a Given Line Through a Point Not on the Given Line.

Given Point P not on \overleftrightarrow{AB} .

Construct $\overleftrightarrow{PE} \perp \overleftrightarrow{AB}$



1. With P as center and any convenient radius, draw an arc that intersects \overleftrightarrow{AB} in two points, C and D .
2. Open the compass to a radius greater than one-half of CD . With C and D as centers, draw intersecting arcs. Label the point of intersection E .
3. Draw \overleftrightarrow{PE} intersecting \overleftrightarrow{AB} at F .



Conclusion $\overleftrightarrow{PE} \perp \overleftrightarrow{AB}$

Proof

Statements	Reasons
1. $\overline{CP} \cong \overline{PD}, \overline{CE} \cong \overline{DE}$	1. Radii of congruent circles are congruent.
2. $CP = PD, CE = DE$	2. Segments that are congruent have the same measure.
3. $\overleftrightarrow{PE} \perp \overline{CD}$	3. If two points are each equidistant from the endpoints of a line segment, then the points determine the perpendicular bisector of the line segment. (Theorem 5.2)
4. $\overleftrightarrow{PE} \perp \overleftrightarrow{AB}$	4. \overline{CD} is a subset of line \overleftrightarrow{AB} . ■

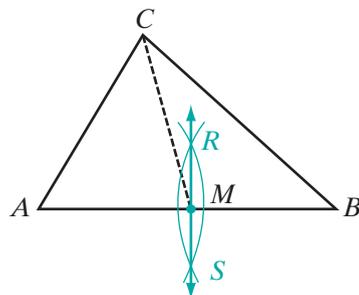
EXAMPLE I

Construct the median to \overline{AB} in $\triangle ABC$.

Construction A median of a triangle is a line segment that joins any vertex of the triangle to the midpoint of the opposite side. To construct the median to \overline{AB} , we must first find the midpoint of \overline{AB} .

1. Construct the perpendicular bisector of \overline{AB} to locate the midpoint. Call the midpoint M .
2. Draw \overline{CM} .

Conclusion \overline{CM} is the median to \overline{AB} in $\triangle ABC$.

**Exercises****Writing About Mathematics**

1. Explain the difference between the altitude of a triangle and the perpendicular bisector of a side of a triangle.
2. Explain how Construction 3 (Construct the perpendicular bisector of a given segment) and Construction 6 (Construct a line perpendicular to a given line through a point not on the given line) are alike and how they are different.

Developing Skills

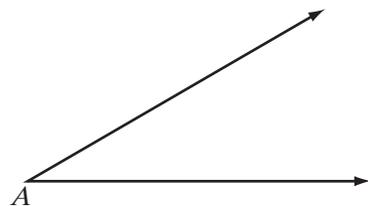
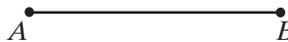
3. Given: \overline{AB}

Construct:

- a. A line segment congruent to \overline{AB} .
 - b. A line segment whose measure is $2AB$.
 - c. The perpendicular bisector of \overline{AB} .
 - d. A line segment whose measure is $1\frac{1}{2}AB$.
4. Given: $\angle A$

Construct:

- a. An angle congruent to $\angle A$.
- b. An angle whose measure is $2m\angle A$.
- c. The bisector of $\angle A$.
- d. An angle whose measure is $2\frac{1}{2}m\angle A$.



5. *Given:* Line segment \overline{ABCD} .

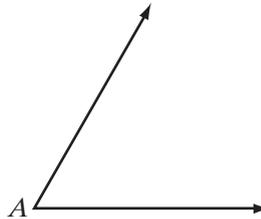


Construct:

- A line segment congruent to \overline{BC} .
 - A triangle with sides congruent to \overline{AB} , \overline{BC} , and \overline{CD} .
 - An isosceles triangle with the base congruent to \overline{AB} and with legs congruent to \overline{BC} .
 - An equilateral triangle with sides congruent to \overline{CD} .
6. *Given:* $\angle A$ with $m\angle A = 60$.

Construct:

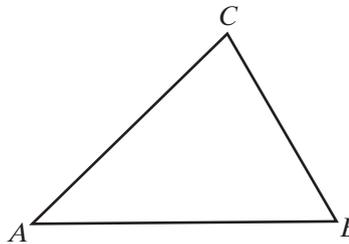
- An angle whose measure is 30.
- An angle whose measure is 15.
- An angle whose measure is 45.



7. *Given:* $\triangle ABC$

Construct:

- The median from vertex C .
 - The altitude to \overline{AB} .
 - The altitude to \overline{BC} .
 - The angle bisector of the triangle from vertex A .
8. **a.** Draw $\triangle ABC$. Construct the three perpendicular bisectors of the sides of $\triangle ABC$. Let P be the point at which the three perpendicular bisectors intersect.
- b.** Is it possible to draw a circle that passes through each of the vertices of the triangle? Explain your answer.



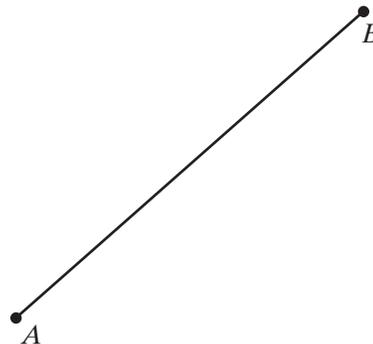
Hands-On Activity



Compass and straightedge constructions can also be done on the computer by using only the point, line segment, line, and circle creation tools of your geometry software and *no* other software tools.

Working with a partner, use either a compass and straightedge, or geometry software to complete the following constructions:

- A square with side \overline{AB} .
- An equilateral triangle with side \overline{AB} .
- 45° angle ABD .
- 30° angle ABD .
- A circle passing through points A , B and C .
(*Hint:* See the proof of Theorem 5.4 or use Theorem 5.3.)



CHAPTER SUMMARY

Definitions to Know

- An **altitude of a triangle** is a line segment drawn from any vertex of the triangle, perpendicular to and ending in the line that contains the opposite side.
- A **median of a triangle** is a line segment that joins any vertex of the triangle to the midpoint of the opposite side.
- An **angle bisector of a triangle** is a line segment that bisects any angle of the triangle and terminates in the side opposite that angle.
- The **perpendicular bisector of a line segment** is a line, a line segment, or a ray that is perpendicular to the line segment at its midpoint.

Postulates 5.1 Radii of congruent circles are congruent.

Theorems and Corollaries

- 5.1 If two sides of a triangle are congruent, the angles opposite these sides are congruent.
- 5.1a The median from the vertex angle of an isosceles triangle bisects the vertex angle.
- 5.1b The median from the vertex angle of an isosceles triangle is perpendicular to the base.
- 5.1c Every equilateral triangle is equiangular.
- 5.2 If two points are each equidistant from the endpoints of a line segment, then the points determine the perpendicular bisector of the line segment.
- 5.3 A point is on the perpendicular bisector of a line segment if and only if it is equidistant from the endpoints of the line segment.
- 5.4 The perpendicular bisectors of the sides of a triangle are concurrent.

VOCABULARY

- 5-1 Altitude of a triangle • Median of a triangle • Angle bisector of a triangle
- 5-3 Isosceles triangle theorem • Corollary
- 5-6 Perpendicular bisector of a line segment • Equidistant • Concurrent • Perpendicular bisector concurrence theorem • Circumcenter
- 5-7 Geometric construction • Straightedge • Compass

REVIEW EXERCISES

1. If $\overleftrightarrow{LMN} \perp \overline{KM}$, $m\angle LMK = x + y$ and $m\angle KMN = 2x - y$, find the value of x and of y .
2. The bisector of $\angle PQR$ in $\triangle PQR$ is \overline{QS} . If $m\angle PQS = x + 20$ and $m\angle SQR = 5x$, find $m\angle PQR$.

3. In $\triangle ABC$, \overline{CD} is both the median and the altitude. If $AB = 5x + 3$, $AC = 2x + 8$, and $BC = 3x + 5$, what is the perimeter of $\triangle ABC$?
4. Angle PQS and angle SQR are a linear pair of angles. If $m\angle PQS = 5a + 15$ and $m\angle SQR = 8a + 35$, find $m\angle PQS$ and $m\angle SQR$.
5. Let D be the point at which the three perpendicular bisectors of the sides of equilateral $\triangle ABC$ intersect. Prove that $\triangle ADB$, $\triangle BDC$, and $\triangle CDA$ are congruent isosceles triangles.
6. Prove that if the median, \overline{CD} , to side \overline{AB} of $\triangle ABC$ is not the altitude to side \overline{AB} , then \overline{AC} is not congruent to \overline{BC} .
7. \overline{AB} is the base of isosceles $\triangle ABC$ and \overline{AB} is also the base of isosceles $\triangle ABD$. Prove that \overleftrightarrow{CD} is the perpendicular bisector of \overline{AB} .
8. In $\triangle ABC$, \overline{CD} is the median to \overline{AB} and $\overline{CD} \cong \overline{DB}$. Prove that $m\angle A + m\angle B = m\angle ACB$. (Hint: Use Theorem 5.1, "If two sides of a triangle are congruent, the angles opposite these sides are congruent.")
9. a. Draw a line, \overleftrightarrow{ADB} . Construct $\overline{CD} \perp \overleftrightarrow{ADB}$.
 b. Use $\angle ADC$ to construct $\angle ADE$ such that $m\angle ADE = 45$.
 c. What is the measure of $\angle EDC$?
 d. What is the measure of $\angle EDB$?
10. a. Draw obtuse $\triangle PQR$ with the obtuse angle at vertex Q .
 b. Construct the altitude from vertex P .

Exploration

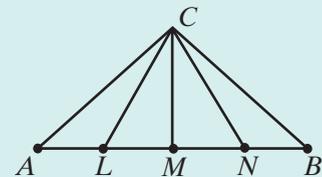
As you have noticed, proofs may be completed using a variety of methods. In this activity, you will explore the reasoning of others.

1. Complete a two-column proof of the following:

Points L , M , and N separate \overline{AB} into four congruent segments. Point C is not on \overline{AB} and \overline{CM} is an altitude of $\triangle CLM$. Prove that $\overline{CA} \cong \overline{CB}$.

2. Cut out each *statement* and each *reason* from your proof, omitting the step numbers.
3. Trade proofs with a partner.
4. Attempt to reassemble your partner's proof.
5. Answer the following questions:

- a. Were you able to reassemble the proof?
- b. Did the reassembled proof match your partner's original proof?
- c. Did you find any components missing or have any components left-over? Why?



CUMULATIVE REVIEW**Chapters 1–5****Part I**

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

- The symbol \overleftrightarrow{ABC} represents
 - a line segment with B between A and C .
 - a line with B the midpoint of \overline{AB} .
 - a line with B between A and C .
 - a ray with endpoint A .
- A triangle with no two sides congruent is
 - a right triangle.
 - an equilateral triangle.
 - an isosceles triangle.
 - a scalene triangle.
- Opposite rays have
 - no points in common.
 - one point in common.
 - two points in common.
 - all points in common.
- The equality $a + 1 = 1 + a$ is an illustration of
 - the commutative property of addition.
 - the additive inverse property.
 - the multiplicative identity property.
 - the closure property of addition.
- The solution set of the equation $1.5x - 7 = 0.25x + 8$ is
 - 120
 - 12
 - 11
 - 1.2
- What is the inverse of the statement “When spiders weave their webs by noon, fine weather is coming soon”?
 - When spiders do not weave their webs by noon, fine weather is not coming soon.
 - When fine weather is coming soon, then spiders weave their webs by noon.
 - When fine weather is not coming soon, spiders do not weave their webs by noon.
 - When spiders weave their webs by noon, fine weather is not coming soon.
- If $\angle ABC$ and $\angle CBD$ are a linear pair of angles, then they must be
 - congruent angles.
 - complementary angles.
 - supplementary angles.
 - vertical angles.

8. Which of the following is *not* an abbreviation for a postulate that is used to prove triangles congruent?
 (1) SSS (2) SAS (3) ASA (4) SSA
9. If the statement “If two angles are right angles, then they are congruent” is true, which of the following statements must also be true?
 (1) If two angles are not right angles, then they are not congruent.
 (2) If two angles are congruent, then they are right angles.
 (3) If two angles are not congruent, then they are not right angles.
 (4) Two angles are congruent only if they are right angles.
10. If \overleftrightarrow{ABC} is a line, which of the following may be false?
 (1) B is on \overline{AC} . (3) B is between A and C .
 (2) $\overline{AB} + \overline{BC} = \overline{AC}$ (4) B is the midpoint of \overline{AC} .

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Angle PQS and angle SQR are a linear pair of angles. If $m\angle PQS = 3a + 18$ and $m\angle SQR = 7a - 2$, find the measure of each angle of the linear pair.
12. Give a reason for each step in the solution of the given equation.

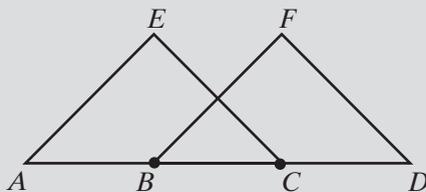
$$\begin{aligned} 5(4 + x) &= 32 - x \\ 20 + 5x &= 32 - x \\ 20 + 5x + x &= 32 - x + x \\ 20 + 6x &= 32 + 0 \\ 20 + 6x &= 32 \\ 6x &= 12 \\ x &= 2 \end{aligned}$$

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

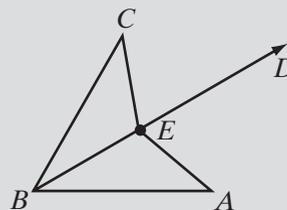
13. Given: $\overline{AE} \cong \overline{BF}$, $\angle ABF$ is the supplement of $\angle A$, and $\overline{AB} \cong \overline{CD}$.

Prove: $\triangle AEC \cong \triangle BFD$



14. Given: $\overline{AB} \cong \overline{CB}$ and E is any point on \overrightarrow{BD} , the bisector of $\angle ABC$.

Prove: $\overline{AE} \cong \overline{CE}$

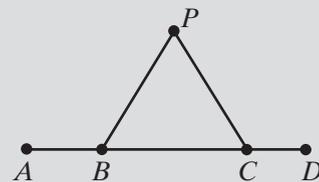


Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Given: Point P is not on \overline{ABCD} and $PB = PC$.

Prove: $\angle ABP \cong \angle DCP$



16. Prove that if \overline{AC} and \overline{BD} are perpendicular bisectors of each other, quadrilateral $ABCD$ is equilateral ($AB = BC = CD = DA$).