CHAPTER

3

PROVING STATEMENTS IN GEOMETRY

After proposing 23 definitions, Euclid listed five postulates and five "common notions." These definitions, postulates, and common notions provided the foundation for the propositions or theorems for which Euclid presented proof. Modern mathematicians have recognized the need for additional postulates to establish a more rigorous foundation for these proofs.

David Hilbert (1862–1943) believed that mathematics should have a logical foundation based on two principles:

1. All mathematics follows from a correctly chosen finite set of assumptions or *axioms*.

2. This set of axioms is not contradictory.

Although mathematicians later discovered that it is not possible to formalize all of mathematics, Hilbert did succeed in putting geometry on a firm logical foundation. In 1899, Hilbert published a text, *Foundations of Geometry*, in which he presented a set of axioms that avoided the limitations of Euclid.

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3-1 INDUCTIVE REASONING

Gina was doing a homework assignment on factoring positive integers. She made a chart showing the number of factors for each of the numbers from 1 to 10. Her chart is shown below.

Number	1	2	3	4	5	6	7	8	9	10
Number of factors	1	2	2	3	2	4	2	4	3	4

She noticed that in her chart, only the perfect square numbers, 1, 4, and 9 had an odd number of factors. The other numbers had an even number of factors. Gina wanted to investigate this observation further so she continued her chart to 25.

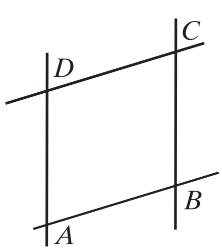
Number	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
Number of factors	2	6	2	4	4	5	2	6	2	6	4	4	2	8	3

Again her chart showed that only the perfect square numbers, 16 and 25, had an odd number of factors and the other numbers had an even number of factors. Gina concluded that this is true for all positive integers. Gina went from a few specific cases to a **generalization**.

Was Gina's conclusion valid? Can you find a counterexample to prove that a perfect square does not always have an odd number of factors? Can you find a non-perfect square that has an odd number of factors?

Scientists perform experiments once, twice, or many times in order to come to a conclusion. A scientist who is searching for a vaccine to prevent a disease will test the vaccine repeatedly to see if it is effective. This method of reasoning, in which a series of particular examples leads to a conclusion, is called **inductive reasoning**.

In geometry, we also perform experiments to discover properties of lines, angles, and polygons and to determine geometric relationships. Most of these experiments involve measurements. Because direct measurements depend on the type of instrument used to measure and the care with which the measurement is made, results can be only approximate. This is the first weakness in attempting to reach conclusions by inductive reasoning.



Use a ruler to draw a pair of parallel line segments by drawing a line segment along opposite edges of the ruler. Turn the ruler and draw another pair of parallel line segments that intersect the first pair. Label the intersections of the line segments A, B, C, and D. The figure, $ABCD$, is a parallelogram. It appears that the opposite sides of the parallelogram have equal measures. Use the ruler to show that this is true.

To convince yourself that this relationship is true in other parallelograms, draw other parallelograms and measure their opposite sides. In each experiment you will find that the opposite sides have the same measure. From these experiments, you can arrive at a general conclusion: The opposite sides of a parallelogram have equal measures. This is an example of inductive reasoning in geometry.

Suppose a student, after examining several parallelograms, made the generalization "All parallelograms have two acute angles and two obtuse angles." Here, a single **counterexample**, such as a parallelogram which is a rectangle and has right angles, is sufficient to show that the general conclusion that the student reached is false.

When we use inductive reasoning, we must use extreme care because we are arriving at a generalization before we have examined every possible example. This is the second weakness in attempting to reach conclusions by inductive reasoning.

When we conduct an experiment we do not give explanations for why things are true. This is the third weakness of inductive reasoning. In light of these weaknesses, when a general conclusion is reached by inductive reasoning alone, it can at best be called probably true. Such statements, that are likely to be true but not yet been proven true by a deductive proof, are called **conjectures**.

Then why study inductive reasoning? Simply because it mimics the way we naturally make new discoveries. Most new knowledge first starts with specific cases, then, through careful study, to a generalization. Only afterwards, is a *proof* or explanation usually sought. Inductive reasoning is therefore a powerful tool in discovering new mathematical facts.

SUMMARY

1. Inductive reasoning is a powerful tool in discovering and making conjectures.

2. Generalizations arising from direct measurements of specific cases are only approximate.

3. Care must be taken when applying inductive reasoning to ensure that all relevant examples are examined (no counterexamples exist).

4. Inductive reasoning does not prove or explain conjectures.

Exercises

Writing About Mathematics

1. After examining several triangles, Mindy concluded that the angles of all triangles are acute. Is Mindy correct? If not, explain to Mindy why she is incorrect.

2. Use a counterexample to show that the whole numbers are not closed under subtraction.

Developing Skills

3. **a.** Using geometry software or a pencil, ruler, and protractor, draw three right triangles that have different sizes and shapes.

b. In each right triangle, measure the two acute angles and find the sum of their measures.

c. Using inductive reasoning based on the experiments just done, make a conjecture about the sum of the measures of the acute angles of a right triangle.

4. **a.** Using geometry software or a pencil, ruler, and protractor, draw three quadrilaterals that have different sizes and shapes.

b. For each quadrilateral, find the midpoint of each side of the quadrilateral, and draw a new quadrilateral using the midpoints as vertices. What appears to be true about the quadrilateral that is formed?

c. Using inductive reasoning based on the experiments just done, make a conjecture about a quadrilateral with vertices at the midpoints of a quadrilateral.

5. **a.** Using geometry software or a pencil, ruler, and protractor, draw three equilateral triangles of different sizes.

b. For each triangle, find the midpoint of each side of the triangle, and draw line segments joining each midpoint. What appears to be true about the four triangles that are formed?

c. Using inductive reasoning based on the experiments just done, make a conjecture about the four triangles formed by joining the midpoints of an equilateral triangle.

In 6–11, describe and perform a series of experiments to investigate whether each statement is probably true or false.

6. If two lines intersect, at least one pair of congruent angles is formed.

7. The sum of the degree measures of the angles of a quadrilateral is 360.

8. If $a^2 < b^2$, then $a < b$.

9. In any parallelogram $ABCD$, $AC = BD$.

10. In any quadrilateral $DEFG$, $\overline{DF}$ bisects $\angle D$ and $\angle F$.

11. The ray that bisects an angle of a triangle intersects a side of the triangle at its midpoint.

12. Adam made the following statement: "For any counting number n, the expression $n^2 + n + 41$ will always be equal to some prime number." He reasoned:

- When $n = 1$, then $n^2 + n + 41 = 1 + 1 + 41 = 43$, a prime number.
- When $n = 2$, then $n^2 + n + 41 = 4 + 2 + 41 = 47$, a prime number.

Use inductive reasoning, by letting n be many different counting numbers, to show that Adam's generalization is probably true, or find a counterexample to show that Adam's generalization is false.

Applying Skills

In 13–16, state in each case whether the conclusion drawn was justified.

13. One day, Joe drove home on Route 110 and found traffic very heavy. He decided never again to drive on this highway on his way home.

14. Julia compared the prices of twenty items listed in the advertising flyers of store A and store B. She found that the prices in store B were consistently higher than those of store A. Julia decided that she will save money by shopping in store A.

15. Tim read a book that was recommended by a friend and found it interesting. He decided that he would enjoy *any* book recommended by that friend.

16. Jill fished in Lake George one day and caught nothing. She decided that there are no fish in Lake George.

17. Nathan filled up his moped's gas tank after driving 92 miles. He concluded that his moped could go at least 92 miles on one tank of gas.

Hands-On Activity

STEP 1. Out of a regular sheet of paper, construct ten cards numbered 1 to 10.

STEP 2. Place the cards face down and in order.

STEP 3. Go through the cards and turn over every card.

STEP 4. Go through the cards and turn over every second card starting with the second card.

STEP 5. Go through the cards and turn over every third card starting with the third card.

STEP 6. Continue this process until you turn over the tenth (last) card.

If you played this game correctly, the cards that are face up when you finish should be 1, 4, and 9.

a. Play this same game with cards numbered 1 to 20. What cards are face up when you finish? What property do the numbers on the cards all have in common?

b. Play this same game with cards numbered 1 to 30. What cards are face up when you finish? What property do the numbers on the cards all have in common?

c. Make a conjecture regarding the numbers on the cards that remain facing up if you play this game with any number of cards.

3-2 DEFINITIONS AS BICONDITIONALS

In mathematics, we often use inductive reasoning to make a conjecture, a statement that appears to be true. Then we use **deductive reasoning** to prove the conjecture. Deductive reasoning uses the laws of logic to combine definitions and general statements that we know to be true to reach a valid conclusion.

Before we discuss this type of reasoning, it will be helpful to review the list of definitions in Chapter 1 given on page 29 that are used in the study of Euclidean geometry.

Definitions and Logic

Each of the definitions given in Chapter 1 can be written in the form of a conditional. For example:

► **A *scalene triangle* is a triangle that has no congruent sides.**

Using the definition of a scalene triangle, we know that:

1. The definition contains a hidden conditional statement and can be rewritten using the words *If . . . then* as follows:

t: A triangle is scalene.
p: A triangle has no congruent sides.
$t \to p$: If a triangle is scalene, then the triangle has no congruent sides.

2. In general, the converse of a true statement is not necessarily true. However, the converse of the conditional form of a definition is always true. For example, the following converse is a true statement:

$p \to t$: If a triangle has no congruent sides, then the triangle is scalene.

3. When a conditional and its converse are both true, the conjunction of these statements can be written as a true biconditional statement. Thus, $(t \to p) \wedge (p \to t)$ is equivalent to the biconditional $(t \leftrightarrow p)$.

Therefore, since both the conditional statement and its converse are true, we can rewrite the above definition as a biconditional statement, using the words *if and only if*, as follows:

► **A triangle is scalene if and only if the triangle has no congruent sides.**

Every good definition can be written as a true biconditional. Definitions will often be used to prove statements in geometry.

EXAMPLE 1

A collinear set of points is a set of points all of which lie on the same straight line.

a. Write the definition in conditional form.

b. Write the converse of the statement given in part **a**.

c. Write the biconditional form of the definition.

Solution **a.** *Conditional:* If a set of points is collinear, then all the points lie on the same straight line.

b. *Converse:* If a set of points all lie on the same straight line, then the set of points is collinear.

c. *Biconditional:* A set of points is collinear if and only if all the points lie on the same straight line.

Exercises

Writing About Mathematics

1. Doug said that "A container is a lunchbox if and only if it can be used to carry food" is not a definition because one part of the biconditional is false. Is Doug correct? If so, give a counterexample to show that Doug is correct.

2. **a.** Give a counterexample to show that the statement "If $\frac{a}{b} < 1$, then $a < b$" is not always true.

b. For what set of numbers is the statement "If $\frac{a}{b} < 1$, then $a < b$" always true?

Developing Skills

In 3–8: **a.** Write each definition in conditional form. **b.** Write the converse of the conditional given in part **a**. **c.** Write the biconditional form of the definition.

3. An equiangular triangle is a triangle that has three congruent angles.

4. The bisector of a line segment is any line, or subset of a line that intersects the segment at its midpoint.

5. An acute angle is an angle whose degree measure is greater than 0 and less than 90.

6. An obtuse triangle is a triangle that has one obtuse angle.

7. A noncollinear set of points is a set of three or more points that do not all lie on the same straight line.

8. A ray is a part of a line that consists of a point on the line, called an endpoint, and all the points on one side of the endpoint.

In 9–14, write the biconditional form of each given definition.

9. A point B is between A and C if A, B, and C are distinct collinear points and $AB + BC = AC$.

10. Congruent segments are segments that have the same length.

11. The midpoint of a line segment is the point of that line segment that divides the segment into two congruent segments.

12. A right triangle is a triangle that has a right angle.

13. A straight angle is an angle that is the union of opposite rays and whose degree measure is 180.

14. Opposite rays are two rays of the same line with a common endpoint and no other point in common.

Applying Skills

In 15–17, write the biconditional form of the definition of each given term.

15. equilateral triangle **16.** congruent angles **17.** perpendicular lines

3-3 DEDUCTIVE REASONING

A **proof** in geometry is a valid argument that establishes the truth of a statement. Most proofs in geometry rely on logic. That is, they are based on a series of statements that are assumed to be true. *Deductive reasoning* uses the laws of logic to link together true statements to arrive at a true conclusion. Since definitions are true statements, they are used in a geometric proof. In the examples that follow, notice how the laws of logic are used in the proofs of geometric statements.

Using Logic to Form a Geometry Proof

Let $\triangle ABC$ be a triangle in which $\overline{AB} \perp \overline{BC}$. We can prove that $\angle ABC$ is a right angle. In this proof, use the following definition:

- Perpendicular lines are two lines that intersect to form right angles.

This definition contains a hidden conditional and can be rewritten as follows:

- *If* two lines are perpendicular, *then* they intersect to form right angles.

Recall that this definition is true for perpendicular line segments as well as for perpendicular lines. Using the specific information cited above, let p represent "$\overline{AB} \perp \overline{BC}$," and let r represent "$\angle ABC$ is a right angle."

The proof is shown by the reasoning that follows:

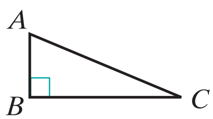
p: $\overline{AB} \perp \overline{BC}$
p is true because it is given.

$p \to r$: If $\overline{AB} \perp \overline{BC}$, then $\angle ABC$ is a right angle.
$p \to r$ is true because it is the definition of perpendicular lines.

r: $\angle ABC$ is a right angle.
r is true by the Law of Detachment.

In the logic-based proof above, notice that the Law of Detachment is cited as a reason for reaching our conclusion. In a typical geometry proof, however, the laws of logic are used to deduce the conclusion but the laws are not listed among the reasons.

Let us restate this proof in the format used often in Euclidean geometry. We write the information known to be true as the "**given**" statements and the conclusion to be proved as the "**prove**". Then we construct a **two-column proof**. In the left column, we write *statements* that we know to be true, and in the right column, we write the *reasons* why each statement is true.



Given: In $\triangle ABC$, $\overline{AB} \perp \overline{BC}$.

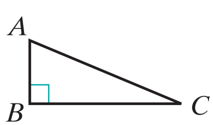
Prove: $\angle ABC$ is a right angle.

Proof:

Statements	Reasons
1. $\overline{AB} \perp \overline{BC}$	**1.** Given.
2. $\angle ABC$ is a right angle.	**2.** If two lines are perpendicular, then they intersect to form right angles.

Notice how the Law of Detachment was used in this geometry proof. By combining statement 1 with reason 2, the conclusion is written as statement 2, just as p and $p \rightarrow r$ led us to the conclusion r using logic. In reason 2, we used the conditional form of the definition of perpendicular lines.

Often in a proof, we find one conclusion in order to use that statement to find another conclusion. For instance, in the following proof, we will begin by proving that $\angle ABC$ is a right angle and then use that fact with the definition of a right triangle to prove that $\triangle ABC$ is a right triangle.



Given: In $\triangle ABC$, $\overline{AB} \perp \overline{BC}$.

Prove: $\triangle ABC$ is a right triangle.

Proof:

Statements	Reasons
1. $\overline{AB} \perp \overline{BC}$	**1.** Given.
2. $\angle ABC$ is a right angle.	**2.** If two lines are perpendicular, then they intersect to form right angles.
3. $\triangle ABC$ is a right triangle.	**3.** If a triangle has a right angle, then it is a right triangle.

Note that in this proof, we used the conditional form of the definition of perpendicular lines in reason 2 and the converse form of the definition of a right triangle in reason 3.

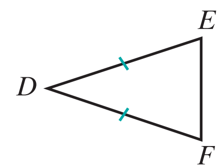
The proof can also be written in paragraph form, also called a **paragraph proof**. Each statement must be justified by stating a definition or another statement that has been accepted or proved to be true. The proof given on page 101 can be written as follows:

Proof: We are given that $\overline{AB} \perp \overline{BC}$. If two lines are perpendicular, then they intersect to form right angles. Therefore, $\angle ABC$ is a right angle. A right triangle is a triangle that has a right angle. Since $\angle ABC$ is an angle of $\triangle ABC$, $\triangle ABC$ is a right triangle.

EXAMPLE 1

Given: $\triangle DEF$ with $DE = DF$

Prove: $\triangle DEF$ is an isosceles triangle.



Proof We will need the following definitions:

- An isosceles triangle is a triangle that has two congruent sides.
- Congruent segments are segments that have the same measure.

We can use these two definitions to first prove that the two segments with equal measure are congruent and then to prove that since the two segments, the sides, are congruent, the triangle is isosceles.

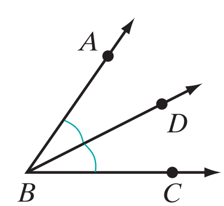
Statements	***Reasons***
1. $DE = DF$	**1.** Given.
2. $\overline{DE} \cong \overline{DF}$	**2.** Congruent segments are segments that have the same measure.
3. $\triangle DEF$ is isosceles.	**3.** An isosceles triangle is a triangle that has two congruent sides.

Alternative Proof The converse of the definition of congruent segments states that if two segments have the same measure, then they are congruent. The given statement, $DE = DF$, means that $\overline{DE}$ and $\overline{DF}$ have the same measure. Therefore, these two sides of $\triangle DEF$ are congruent. The converse of the definition of an isosceles triangle states that if a triangle has two congruent sides, then it is isosceles. Therefore, since $\overline{DE}$ and $\overline{DF}$ are congruent sides of $\triangle DEF$, $\triangle DEF$ is isosceles.

EXAMPLE 2

Given: $\overrightarrow{BD}$ is the bisector of $\angle ABC$.

Prove: $m\angle ABD = m\angle DBC$



Proof We will need the following definitions:

- The bisector of an angle is a ray whose endpoint is the vertex of the angle and that divides the angle into two congruent angles.
- Congruent angles are angles that have the same measure.

First use the definition of an angle bisector to prove that the angles are congruent. Then use the definition of congruent angles to prove that the angles have equal measures.

Statements	***Reasons***
1. $\overrightarrow{BD}$ is the bisector of $\angle ABC$.	**1.** Given.
2. $\angle ABD \cong \angle DBC$	**2.** The bisector of an angle is a ray whose endpoint is the vertex of the angle and that divides the angle into two congruent angles.
3. $m\angle ABD = m\angle DBC$	**3.** Congruent angles are angles that have the same measure.

Exercises

Writing About Mathematics

1. Is an equilateral triangle an isosceles triangle? Justify your answer.

2. Is it possible that the points A, B, and C are collinear but $AB + BC \neq AC$? Justify your answer.

Developing Skills

In 3–6, in each case: **a.** Draw a diagram to illustrate the given statement. **b.** Write a definition or definitions from geometry, in conditional form, that can be used with the *given* statement to justify the conclusion.

3. *Given:* $\overrightarrow{SP}$ bisects $\angle RST$.
Conclusion: $\angle RSP \cong \angle PST$

4. *Given:* $\triangle ABC$ is a scalene triangle.
Conclusion: $AB \neq BC$

5. *Given:* $\overleftrightarrow{BCD} \perp \overleftrightarrow{ACE}$
Conclusion: $m\angle ACD = 90$

6. *Given:* $AB + BC = AC$ with $\overline{ABC}$
Conclusion: B is between A and C.

In 7–12, in each case: **a.** Draw a diagram to illustrate the given statement. **b.** Write a definition from geometry, in conditional form, that can be used with the given statement to make a conclusion. **c.** From the given statement and the definition that you chose, draw a conclusion.

7. *Given:* $\triangle LMN$ with $\overline{LM} \perp \overline{MN}$

8. *Given:* $\overleftrightarrow{AB}$ bisects $\overline{DE}$ at F.

9. *Given:* $PQ + QR = PR$ with $\overline{PQR}$

10. *Given:* $\overrightarrow{ST}$ and $\overrightarrow{SR}$ are opposite rays.

11. *Given:* M is the midpoint of $\overline{LN}$.

12. *Given:* $0 < m\angle A < 90$

Applying Skills

In 13–15: **a.** Give the reason for each statement of the proof. **b.** Write the proof in paragraph form.

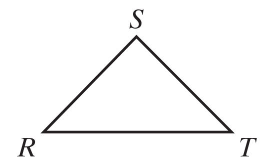
13. *Given:* M is the midpoint of $\overline{AMB}$.

Prove: $AM = MB$

Statements	Reasons
1. M is the midpoint of $\overline{AMB}$.	**1.** ________
2. $\overline{AM} \cong \overline{MB}$	**2.** ________
3. $AM = MB$	**3.** ________

14. *Given:* $\triangle RST$ with $RS = ST$.

Prove: $\triangle RST$ is an isosceles triangle.

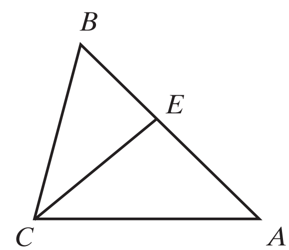


Statements	Reasons
1. $RS = ST$	**1.** ________
2. $\overline{RS} \cong \overline{ST}$	**2.** ________
3. $\triangle RST$ is isosceles.	**3.** ________

15. *Given:* In $\triangle ABC$, $\overrightarrow{CE}$ bisects $\angle ACB$.

Prove: $m\angle ACE = m\angle BCE$

Statements	Reasons
1. $\overrightarrow{CE}$ bisects $\angle ACB$.	**1.** ________________
2. $\angle ACE \cong \angle ECB$	**2.** ________________
3. $m\angle ACE = m\angle ECB$	**3.** ________________



16. Complete the following proof by writing the statement for each step.

Given: $\overline{DEF}$ with $DE = EF$.

Prove: E is the midpoint of $\overline{DEF}$.

Statements	Reasons
1. ________________________	**1.** Given.
2. ________________________	**2.** Congruent segments are segments that have the same measure.
3. ________________________	**3.** The midpoint of a line segment is the point of that line segment that divides the segment into congruent segments.

17. In $\triangle ABC$, $m\angle A < 90$ and $m\angle B < 90$. If $\triangle ABC$ is an obtuse triangle, why is $m\angle C > 90$? Justify your answer with a definition.

18. Explain why the following proof is incorrect.

Given: Isosceles $\triangle ABC$ with $\angle A$ as the vertex angle.

Prove: $BC = AC$

Statements	Reasons
1. $\triangle ABC$ is isosceles.	**1.** Given.
2. $\overline{BC} \cong \overline{AC}$	**2.** An isosceles triangle has two congruent sides.
3. $BC = AC$	**3.** Congruent segments are segments that have the same measure.

3-4 DIRECT AND INDIRECT PROOFS

A proof that starts with the given statements and uses the laws of logic to arrive at the statement to be proved is called a **direct proof**. A proof that starts with the *negation* of the statement to be proved and uses the laws of logic to show that it is false is called an **indirect proof** or a **proof by contradiction**.

An indirect proof works because when the negation of a statement is false, the statement must be true. Therefore, if we can show that the negation of the statement to be proved is false, then we can conclude that the statement is true.

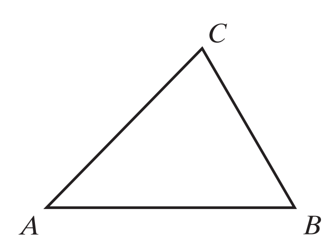
Direct Proof

All of the proofs in Section 3-3 are direct proofs. In most direct proofs we use definitions together with the Law of Detachment to arrive at the desired conclusion. Example 1 uses direct proof.

EXAMPLE 1

Given: $\triangle ABC$ is an acute triangle.

Prove: $m\angle A < 90$

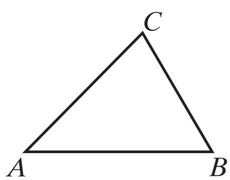


In this proof, we will use the following definitions:

- An acute triangle is a triangle that has three acute angles.
- An acute angle is an angle whose degree measure is greater than 0 and less than 90.

In the proof, we will use the conditional form of these definitions.

Proof



Statements	Reasons
1. $\triangle ABC$ is an acute triangle.	**1.** Given.
2. $\angle A$, $\angle B$, and $\angle C$ are all acute.	**2.** If a triangle is acute, then the triangle has three acute angles.
3. $m\angle A < 90$	**3.** If an angle is acute, then its degree measure is greater than 0 and less than 90.

Note: In this proof, the "and" statement is important for the conclusion. In statement 2, the conjunction can be rewritten as "$\angle A$ is acute, and $\angle B$ is acute, and $\angle C$ is acute." We know from logic that when a conjunction is true, each conjunct is true. Also, in Reason 3, the conclusion of the conditional is a conjunction: "The degree measure is greater than 0 *and* the degree measure is less than 90." Again, since this conjunction is true, each conjunct is true.

Indirect Proof

In an indirect proof, let p be the given and q be the conclusion. Take the following steps to show that the conclusion is true:

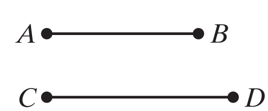
1. Assume that the negation of the conclusion is true.

2. Use this assumption to arrive at a statement that contradicts the given statement or a true statement derived from the given statement.

3. Since the assumption leads to a contradiction, it must be false. The negation of the assumption, the desired conclusion, must be true.

Let us use an indirect proof to prove the following statement: If the measures of two segments are unequal, then the segments are not congruent.

EXAMPLE 2



Given: $\overline{AB}$ and $\overline{CD}$ such that $AB \neq CD$.

Prove: $\overline{AB}$ and $\overline{CD}$ are not congruent segments

Proof Start with an assumption that is the negation of the conclusion.

Statements	*Reasons*
1. $\overline{AB}$ and $\overline{CD}$ are congruent segments.	**1.** Assumption.
2. $AB = CD$	**2.** Congruent segments are segments that have the same measure.
3. $AB \neq CD$	**3.** Given.
4. $\overline{AB}$ and $\overline{CD}$ are not congruent segments.	**4.** Contradiction in 2 and 3. Therefore, the assumption is false and its negation is true.

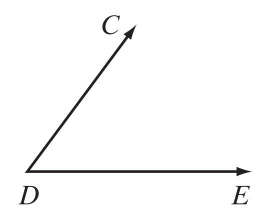
In this proof, and most indirect proofs, our reasoning reflects the contrapositive of a definition. For example:

Definition: Congruent segments are segments that have the same measure.
Conditional: If segments are congruent, then they have the same measure.
Contrapositive: If segments do not have the same measure, then they are not congruent.

Note: To learn how the different methods of proof work, you will be asked to prove some simple statements both directly and indirectly in this section. Thereafter, you should use the method that seems more efficient, which is usually a direct proof. However, in some cases, only an indirect proof is possible.

EXAMPLE 3

Write a direct and an indirect proof of the following:



Given: $m\angle CDE \neq 90$

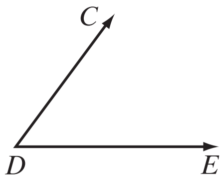
Prove: $\overline{CD}$ is not perpendicular to $\overline{DE}$.

In this proof, we will use two definitions:

- If an angle is a right angle, then its degree measure is 90.
- If two intersecting lines are perpendicular, then they form right angles.

Direct Proof We will use the contrapositive form of the two definitions:

- If the degree measure of an angle is not 90, then it is not a right angle.
- If two intersecting lines do not form right angles, then they are not perpendicular.



Statements	Reasons
1. m$\angle CDE \neq 90$	**1.** Given.
2. $\angle CDE$ is not a right angle.	**2.** If the degree measure of an angle is not 90, then it is not a right angle.
3. $\overline{CD}$ is not perpendicular to $\overline{DE}$.	**3.** If two intersecting lines do not form right angles, then they are not perpendicular. ☐

Indirect Proof We will use the negation of the statement that is to be proved as an assumption, and then arrive at a contradiction using the conditional form of the two definitions.

Statements	Reasons
1. $\overline{CD}$ is perpendicular to $\overline{DE}$.	**1.** Assumption.
2. $\angle CDE$ is a right angle.	**2.** If two intersecting lines are perpendicular, then they form right angles.
3. m$\angle CDE = 90$	**3.** If an angle is a right angle, then its degree measure is 90.
4. m$\angle CDE \neq 90$.	**4.** Given.
5. $\overline{CD}$ is not perpendicular to $\overline{DE}$.	**5.** Contradiction in 3 and 4. Therefore, the assumption is false and its negation is true. ☐

Exercises

Writing About Mathematics

1. If we are given $\angle ABC$, is it true that $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{BC}$? Explain why or why not.

2. Glen said that if we are given line $\overleftrightarrow{ABC}$, then we know that A, B, and C are collinear and B is between A and C. Do you agree with Glen? Justify your answer.

Developing Skills

In 3–8: **a.** Draw a figure that represents the statement to be proved. **b.** Write a direct proof in two-column form. **c.** Write an indirect proof in two-column form.

3. *Given:* $LM = MN$

Prove: $\overline{LM} \cong \overline{MN}$

4. *Given:* $\angle PQR$ is a straight angle.

Prove: $m\angle PQR = 180$

5. *Given:* $\angle PQR$ is a straight angle.

Prove: $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are opposite rays.

6. *Given:* $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are opposite rays.

Prove: P, Q, and R are on the same line.

7. *Given:* $\angle PQR$ is a straight angle.

Prove: P, Q, and R are on the same line.

8. *Given:* $\overrightarrow{EG}$ bisects $\angle DEF$.

Prove: $m\angle DEG = m\angle GEF$

9. Compare the direct proofs to the indirect proofs in problems 3–8. In these examples, which proofs were longer? Why do you think this is the case?

10. Draw a figure that represents the statement to be proved and write an indirect proof:

Given: $\overrightarrow{EG}$ does not bisect $\angle DEF$.

Prove: $\angle DEG$ is not congruent to $\angle GEF$.

Applying Skills

11. In order to prove a conditional statement, we let the hypothesis be the *given* and the conclusion be the *prove*.

If $\overrightarrow{BD}$ is perpendicular to $\overleftrightarrow{ABC}$, then $\overrightarrow{BD}$ is the bisector of $\angle ABC$.

a. Write the hypothesis of the conditional as the *given*.

b. Write the conclusion of the conditional as the *prove*.

c. Write a direct proof for the conditional.

12. If $m\angle EFG \neq 180$, then $\overrightarrow{FE}$ and $\overrightarrow{FG}$ are not opposite rays.

a. Write the hypothesis of the conditional as the *given*.

b. Write the conclusion of the conditional as the *prove*.

c. Write an indirect proof for the conditional.

3-5 POSTULATES, THEOREMS, AND PROOF

A valid argument that leads to a true conclusion begins with true statements. In Chapter 1, we listed undefined terms and definitions that we accept as being true. We have used the undefined terms and definitions to draw conclusions.

At times, statements are made in geometry that are neither undefined terms nor definitions, and yet we know these are true statements. Some of these statements seem so "obvious" that we accept them without proof. Such a statement is called a **postulate** or an **axiom**.

Some mathematicians use the term "axiom" for a general statement whose truth is assumed without proof, and the term "postulate" for a geometric statement whose truth is assumed without proof. We will use the term "postulate" for both types of assumptions.

DEFINITION

A **postulate** is a statement whose truth is accepted without proof.

When we apply the laws of logic to definitions and postulates, we are able to prove other statements. These statements are called **theorems**.

DEFINITION

A **theorem** is a statement that is proved by deductive reasoning.

The entire body of knowledge that we know as geometry consists of undefined terms, defined terms, postulates, and theorems which we use to prove other theorems and to justify applications of these theorems.

The First Postulates Used in Proving Statements

Geometry is often concerned with measurement. In Chapter 1 we listed the properties of the number system. These properties include closure, the commutative, associative, inverse, identity, and distributive properties of addition and multiplication and the multiplication property of zero. We will use these properties as postulates in geometric proof. Other properties that we will use as postulates are the properties of equality.

When we state the relation "x is equal to y," symbolized as "$x = y$," we mean that x and y are two different names for the same element of a set, usually a number. For example:

1. When we write $AB = CD$, we mean that length of $\overline{AB}$ and the length of $\overline{CD}$ are the same number.
2. When we write $m\angle P = m\angle N$, we mean that $\angle P$ and $\angle N$ contain the same number of degrees.

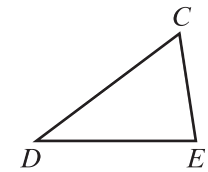
Many of our definitions, for example, congruence, midpoint, and bisector, state that two measures are equal. There are three basic properties of equality.

The Reflexive Property of Equality: $a = a$

The **reflexive property of equality** is stated in words as follows:

Postulate 3.1 A quantity is equal to itself.

For example, in $\triangle CDE$, observe that:



1. The length of a segment is equal to itself:

$$CD = CD \qquad DE = DE \qquad EC = EC$$

2. The measure of an angle is equal to itself:

$$m\angle C = m\angle C \qquad m\angle D = m\angle D \qquad m\angle E = m\angle E$$

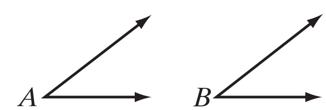
The Symmetric Property of Equality: If $a = b$, then $b = a$.

The **symmetric property of equality** is stated in words as follows:

Postulate 3.2

> An equality may be expressed in either order.

For example:



1. If $LM = NP$, then $NP = LM$.
2. If $m\angle A = m\angle B$, then $m\angle B = m\angle A$.

The Transitive Property of Equality: If $a = b$ and $b = c$, then $a = c$.

This property states that, if a and b have the same value, and b and c have the same value, it follows that a and c have the same value. The **transitive property of equality** is stated in words as follows:

Postulate 3.3

> Quantities equal to the same quantity are equal to each other.

The lengths or measures of segments and angles are numbers. In the set of real numbers, the relation "is equal to" is said to be reflexive, symmetric, and transitive. A relation for which these postulates are true is said to be an **equivalence relation**.

Congruent segments are segments with equal measures and congruent angles are angles with equal measures. This suggests that "is congruent to" is also an equivalence relation for the set of line segments. For example:

1. $\overline{AB} \cong \overline{AB}$
   A line segment is congruent to itself.
2. If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.
   Congruence can be stated in either order.
3. If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.
   Segments congruent to the same segment are congruent to each other.

Therefore, we can say that "is congruent to" is an equivalence relation on the set of line segments.

We will use these postulates of equality in deductive reasoning. In constructing a valid proof, we follow these steps:

1. A diagram is used to visualize what is known and how it relates to what is to be proved.
2. State the hypothesis or premise as the *given*, in terms of the points and lines in the diagram. The premises are the given facts.
3. The conclusion contains what is to be proved. State the conclusion as the *prove*, in terms of the points and lines in the diagram.
4. We present the *proof*, the deductive reasoning, as a series of statements. Each statement in the proof should be justified by the given, a definition, a postulate, or a previously proven theorem.

EXAMPLE 1

If $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$, then $AB = CD$.

Given: $\overline{AB} \cong \overline{BC}$ and $\overline{BC} \cong \overline{CD}$

Prove: $AB = CD$

Proof

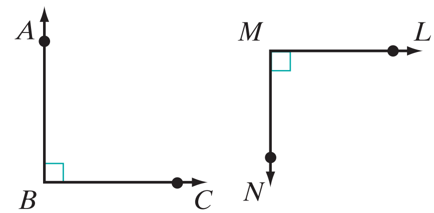
Statements	Reasons
1. $\overline{AB} \cong \overline{BC}$	**1.** Given.
2. $AB = BC$	**2.** Congruent segments are segments that have the same measure.
3. $\overline{BC} \cong \overline{CD}$	**3.** Given.
4. $BC = CD$	**4.** Congruent segments are segments that have the same measure.
5. $AB = CD$	**5.** Transitive property of equality (steps 2 and 4).

Alternative Proof

Statements	Reasons
1. $\overline{AB} \cong \overline{BC}$	**1.** Given.
2. $\overline{BC} \cong \overline{CD}$	**2.** Given.
3. $\overline{AB} \cong \overline{CD}$	**3.** Transitive property of congruence.
4. $AB = CD$	**4.** Congruent segments are segments that have the same measure.

EXAMPLE 2

If $\overline{AB} \perp \overline{BC}$ and $\overline{LM} \perp \overline{MN}$, then $m\angle ABC = m\angle LMN$.



Given: $\overline{AB} \perp \overline{BC}$ and $\overline{LM} \perp \overline{MN}$

Prove: $m\angle ABC = m\angle LMN$

Proof

Statements	Reasons
1. $\overline{AB} \perp \overline{BC}$	**1.** Given.
2. $\angle ABC$ is a right angle.	**2.** Perpendicular lines are two lines that intersect to form right angles.
3. $m\angle ABC = 90$	**3.** A right angle is an angle whose degree measure is 90.
4. $\overline{LM} \perp \overline{MN}$	**4.** Given.
5. $\angle LMN$ is a right angle.	**5.** Perpendicular lines are two lines that intersect to form right angles.
6. $m\angle LMN = 90$	**6.** A right angle is an angle whose degree measure is 90.
7. $90 = m\angle LMN$	**7.** Symmetric property of equality.
8. $m\angle ABC = m\angle LMN$	**8.** Transitive property of equality (steps 3 and 7).

Exercises

Writing About Mathematics

1. Is "is congruent to" an equivalence relation for the set of angles? Justify your answer.

2. Is "is perpendicular to" an equivalence relation for the set of lines? Justify your answer.

Developing Skills

In 3–6, in each case: state the postulate that can be used to show that each conclusion is valid.

3. $CD = CD$

4. $2 + 3 = 5$ and $5 = 1 + 4$. Therefore, $2 + 3 = 4 + 1$.

5. $10 = a + 7$. Therefore $a + 7 = 10$.

6. $m\angle A = 30$ and $m\angle B = 30$. Therefore, $m\angle A = m\angle B$.

Applying Skills

In 7–10, write the reason of each step of the proof.

7. *Given:* $y = x + 4$ and $y = 7$

Prove: $x + 4 = 7$

Statements	Reasons
1. $y = x + 4$	**1.** ________
2. $x + 4 = y$	**2.** ________
3. $y = 7$	**3.** ________
4. $x + 4 = 7$	**4.** ________

8. *Given:* $AB + BC = AC$ and $AB + BC = 12$

Prove: $AC = 12$

Statements	Reasons
1. $AB + BC = AC$	**1.** ________
2. $AC = AB + BC$	**2.** ________
3. $AB + BC = 12$	**3.** ________
4. $AC = 12$	**4.** ________

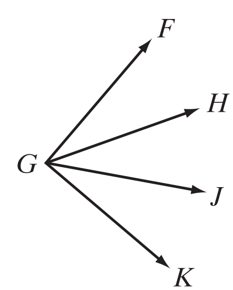
9. *Given:* M is the midpoint of $\overline{LN}$ and N is the midpoint of $\overline{MP}$.

Prove: $LM = NP$

Statements	Reasons
1. M is the midpoint of $\overline{LN}$.	**1.** ________
2. $\overline{LM} \cong \overline{MN}$	**2.** ________
3. $LM = MN$	**3.** ________
4. N is the midpoint of $\overline{MP}$.	**4.** ________
5. $\overline{MN} \cong \overline{NP}$	**5.** ________
6. $MN = NP$	**6.** ________
7. $LM = NP$	**7.** ________

10. *Given:* $m\angle FGH = m\angle JGK$ and $m\angle HGJ = m\angle JGK$

Prove: $\overrightarrow{GH}$ is the bisector of $\angle FGJ$.

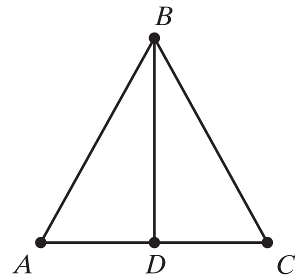


Statements	Reasons
1. $m\angle FGH = m\angle JGK$	**1.** ________
2. $m\angle HGJ = m\angle JGK$	**2.** ________
3. $m\angle FGH = m\angle HGJ$	**3.** ________
4. $\angle FGH \cong \angle HGJ$	**4.** ________
5. $\overrightarrow{GH}$ is the bisector of $\angle FGJ$.	**5.** ________

11. Explain why the following proof is incorrect.

Given: $\triangle ABC$ with D a point on $\overline{BC}$.

Prove: $\angle ADB \cong \angle ADC$



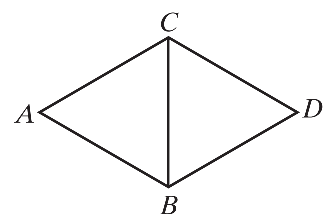
Statements	***Reasons***
1. $\angle ADB$ and $\angle ADC$ are right angles.	**1.** Given (from the diagram).
2. $m\angle ADB = 90$ and $m\angle ADC = 90$.	**2.** A right angle is an angle whose degree measure is 90.
3. $m\angle ADB = m\angle ADC$	**3.** Transitive property of equality.
4. $\angle ADB \cong \angle ADC$	**4.** If two angles have the same measure, then they are congruent.

Hands-On Activity

Working with a partner: **a.** Determine the definitions and postulates that can be used with the given statement to write a proof. **b.** Write a proof in two-column form.

Given: $\triangle ABC$ and $\triangle BCD$ are equilateral.

Prove: $AB = CD$



3-6 THE SUBSTITUTION POSTULATE

The substitution postulate allows us to replace one quantity, number, or measure with its equal. The **substitution postulate** is stated in words as follows:

Postulate 3.4 A quantity may be substituted for its equal in any statement of equality.

From $x = y$ and $y = 8$, we can conclude that $x = 8$. This is an example of the transitive property of equality, but we can also say that we have used the substitution property by substituting 8 for y in the equation $x = y$.

From $y = x + 7$ and $x = 3$, we can conclude that $y = 3 + 7$. This is not an example of the transitive property of equality. We have used the substitution property to replace x with its equal, 3, in the equation $y = x + 7$.

We use this postulate frequently in algebra. For example, if we find the solution of an equation, we can substitute that solution to show that we have a true statement.

$$4x - 1 = 3x + 7$$
$$x = 8$$

Check

$$4x - 1 = 3x + 7$$
$$4(8) - 1 \stackrel{?}{=} 3(8) + 7$$
$$31 = 31 \checkmark$$

In a system of two equations in two variables, x and y, we can solve one equation for y and substitute in the other equation. For example, we are using the substitution postulate in the third line of this solution:

$$3x + 2y = 13$$
$$y = x - 1$$

$$3x + 2(x - 1) = 13$$
$$3x + 2x - 2 = 13$$
$$5x = 15$$
$$x = 3$$
$$y = 3 - 1 = 2$$

EXAMPLE 1

Given: $CE = 2CD$ and $CD = DE$

Prove: $CE = 2DE$

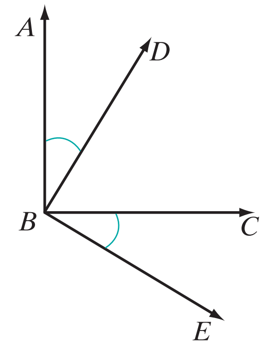
Proof

Statements	***Reasons***
1. $CE = 2CD$	**1.** Given.
2. $CD = DE$	**2.** Given.
3. $CE = 2DE$	**3.** Substitution postulate. (*Or:* A quantity may be substituted for its equal in any expression of equality.)

EXAMPLE 2

Given: $m\angle ABD + m\angle DBC = 90$ and
$m\angle ABD = m\angle CBE$

Prove: $m\angle CBE + m\angle DBC = 90$



Proof

Statements	Reasons
1. $m\angle ABD + m\angle DBC = 90$	**1.** Given.
2. $m\angle ABD = m\angle CBE$	**2.** Given.
3. $m\angle CBE + m\angle DBC = 90$	**3.** Substitution postulate.

Exercises

Writing About Mathematics

1. If we know that $\overline{PQ} \cong \overline{RS}$ and that $\overline{RS} \cong \overline{ST}$, can we conclude that $\overline{PQ} \cong \overline{ST}$? Justify your answer.

2. If we know that $\overline{PQ} \cong \overline{RS}$, and that $\overline{RS} \perp \overline{ST}$, can we use the substitution postulate to conclude that $\overline{PQ} \perp \overline{ST}$? Justify your answer.

Developing Skills

In 3–10, in each case write a proof giving the reason for each statement in your proof.

3. *Given:* $MT = \frac{1}{2}RT$ and $RM = MT$

Prove: $RM = \frac{1}{2}RT$

4. *Given:* $AD + DE = AE$ and $AD = EB$

Prove: $EB + DE = AE$

5. *Given:* $m\angle a + m\angle b = 180$ and
$m\angle a = m\angle c$

Prove: $m\angle c + m\angle b = 180$

6. *Given:* $y = x + 5$ and $y + 7 = 2x$

Prove: $x + 5 + 7 = 2x$

7. *Given:* $12 = x + y$ and $x = 8$

Prove: $12 = 8 + y$

8. *Given:* $BC^2 = AB^2 + AC^2$ and
$AB = DE$

Prove: $BC^2 = DE^2 + AC^2$

9. *Given:* $AB = \sqrt{CD}$, $\frac{1}{2}GH = EF$, and
$\sqrt{CD} = EF$

Prove: $AB = \frac{1}{2}GH$

10. *Given:* $m\angle Q + m\angle R + m\angle S = 75$,
$m\angle Q + m\angle S = m\angle T$, and
$m\angle R + m\angle T = m\angle U$

Prove: $m\angle U = 75$

3-7 THE ADDITION AND SUBTRACTION POSTULATES

The Partition Postulate

When three points, A, B, and C, lie on the same line, the symbol $\overleftrightarrow{ABC}$ is a way of indicating the following equivalent facts about these points:

- B is on the line segment $\overline{AC}$.
- B is between A and C.
- $\overline{AB} + \overline{BC} = \overline{AC}$

Since A, B, and C lie on the same line, we can also conclude that $AB + BC = AC$. In other words, B separates $\overline{AC}$ into two segments whose sum is $\overline{AC}$. This fact is expressed in the following postulate called the **partition postulate**.

Postulate 3.5 A whole is equal to the sum of all its parts.

This postulate applies to any number of segments or to their lengths.

- If B is between A and C, then A, B, and C are collinear.

$$AB + BC = AC$$
$$\overline{AB} + \overline{BC} = \overline{AC}$$

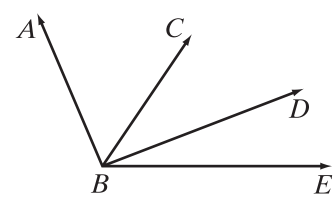
- If C is between B and D, then B, C, and D are collinear.

$$BC + CD = BD$$
$$\overline{BC} + \overline{CD} = \overline{BD}$$

- If A, B, and C are collinear and B, C, and D are collinear, then A, B, C, and D are collinear, that is, $\overleftrightarrow{ABCD}$. We may conclude:

$$AB + BC + CD = AD$$
$$\overline{AB} + \overline{BC} + \overline{CD} = \overline{AD}$$

This postulate also applies to any number of angles or their measures.



- If $\overrightarrow{BC}$ is a ray in the interior of $\angle ABD$:

$$m\angle ABC + m\angle CBD = m\angle ABD$$
$$\angle ABC + \angle CBD = \angle ABD$$

- If $\overrightarrow{BD}$ is a ray in the interior of $\angle CBE$:

$$m\angle CBD + m\angle DBE = m\angle CBE$$
$$\angle CBD + \angle DBE = \angle CBE$$

- We can also conclude that:

$$m\angle ABC + m\angle CBD + m\angle DBE = m\angle ABE$$
$$\angle ABC + \angle CBD + \angle DBE = \angle ABE$$

Note: We write $\angle x + \angle y$ to represent the sum of the angles, $\angle x$ and $\angle y$, only if $\angle x$ and $\angle y$ are adjacent angles.

Since $AB = DE$ indicates that $\overline{AB} \cong \overline{DE}$ and $m\angle ABC = m\angle DEF$ indicates that $\angle ABC \cong \angle DEF$, that is, since equality implies congruency, we can restate the partition postulate in terms of congruent segments and angles.

Postulate 3.5.1 A segment is congruent to the sum of all its parts.

Postulate 3.5.2 An angle is congruent to the sum of all its parts.

The Addition Postulate

The **addition postulate** may be stated in symbols or in words as follows:

Postulate 3.6 If $a = b$ and $c = d$, then $a + c = b + d$.

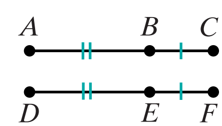
If equal quantities are added to equal quantities, the sums are equal.

The following example proof uses the addition postulate.

EXAMPLE 1

Given: $\overleftrightarrow{ABC}$ and $\overleftrightarrow{DEF}$ with $AB = DE$ and $BC = EF$.

Prove: $AC = DF$



Proof

Statements	***Reasons***
1. $AB = DE$	**1.** Given.
2. $BC = EF$	**2.** Given.
3. $AB + BC = DE + EF$	**3.** Addition postulate. (*Or:* If equal quantities are added to equal quantities, the sums are equal.)
4. $AB + BC = AC$ $DE + EF = DF$	**4.** Partition postulate. (*Or:* A whole is equal to the sum of all its parts.)
5. $AC = DF$	**5.** Substitution postulate.

Just as the partition postulate was restated for congruent segments and congruent angles, so too can the addition postulate be restated in terms of congruent segments and congruent angles. Recall that we can add line segments $\overline{AB}$ and $\overline{BC}$ if and only if B is between A and C.

Postulate 3.6.1

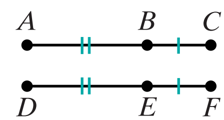
> If congruent segments are added to congruent segments, the sums are congruent.

The example proof just demonstrated can be rewritten in terms of the congruence of segments.

EXAMPLE 2

Given: $\overleftrightarrow{ABC}$ and $\overleftrightarrow{DEF}$ with $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$.

Prove: $\overline{AC} \cong \overline{DF}$



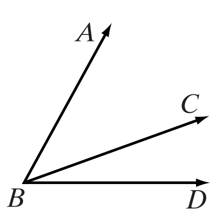
Proof

Statements	Reasons
1. $\overline{AB} \cong \overline{DE}$	**1.** Given.
2. $\overline{BC} \cong \overline{EF}$	**2.** Given.
3. $\overline{AB} + \overline{BC} \cong \overline{DE} + \overline{EF}$	**3.** Addition postulate. (*Or:* If congruent segments are added to congruent segments, the sums are congruent segments.)
4. $\overline{AB} + \overline{BC} = \overline{AC}$	**4.** Partition postulate.
5. $\overline{DE} + \overline{EF} = \overline{DF}$	**5.** Partition postulate.
6. $\overline{AC} \cong \overline{DF}$	**6.** Substitution postulate (steps 3, 4, 5).

When the addition postulate is stated for congruent angles, it is called the **angle addition postulate**:

Postulate 3.6.2

> If congruent angles are added to congruent angles, the sums are congruent.



Recall that to add angles, the angles must have a common endpoint, a common side between them, and no common interior points.

In the diagram, $\angle ABC + \angle CBD \cong \angle ABD$.

The Subtraction Postulate

The **subtraction postulate** may also be stated in symbols or in words.

Postulate 3.7

> If $a = b$, and $c = d$, then $a - c = b - d$.
>
> If equal quantities are subtracted from equal quantities, the differences are equal.

Just as the addition postulate was restated for congruent segments and congruent angles, so too may we restate the subtraction postulate in terms of congruent segments and congruent angles.

Postulate 3.7.1

> If congruent segments are subtracted from congruent segments, the differences are congruent.

Postulate 3.7.2

> If congruent angles are subtracted from congruent angles, the differences are congruent.

In Example 3, equal numbers are subtracted. In Example 4, congruent lengths are subtracted.

EXAMPLE 3

Given: $x + 6 = 14$

Prove: $x = 8$

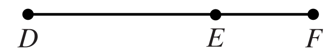
Proof

Statements	***Reasons***
1. $x + 6 = 14$	**1.** Given.
2. $6 = 6$	**2.** Reflexive property.
3. $x = 8$	**3.** Subtraction postulate.

EXAMPLE 4

Given: $\overline{DEF}$, E is between D and F

Prove: $\overline{DE} \cong \overline{DF} - \overline{EF}$



Proof

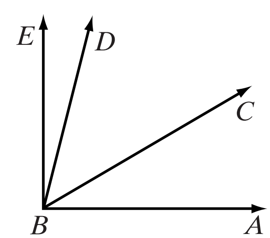
Statements	Reasons
1. E is between D and F.	**1.** Given.
2. $\overline{DE} + \overline{EF} \cong \overline{DF}$	**2.** Partition postulate.
3. $\overline{EF} \cong \overline{EF}$	**3.** Reflexive property.
4. $\overline{DE} \cong \overline{DF} - \overline{EF}$	**4.** Subtraction postulate.

Exercises

Writing About Mathematics

1. Cassie said that we do not need the subtraction postulate because subtraction can be expressed in terms of addition. Do you agree with Cassie? Explain why or why not.

2. In the diagram, $m\angle ABC = 30$, $m\angle CBD = 45$, and $m\angle DBE = 15$.



a. Does $m\angle CBD = m\angle ABC + m\angle DBE$? Justify your answer.

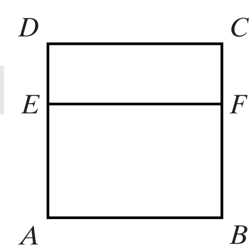
b. Is $\angle CBD \cong \angle ABC + \angle DBE$? Justify your answer.

Developing Skills

In 3 and 4, in each case fill in the missing *statement* or *reason* in the proof to show that the conclusion is valid.

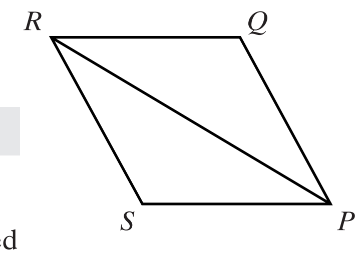
3. *Given:* $\overline{AED}$ and $\overline{BFC}$, $AE = BF$, and $ED = FC$

Prove: $AD = BC$



Statements	Reasons
1. $\overline{AED}$ and $\overline{BFC}$	**1.** Given.
2. $AE + ED = AD$ $BF + FC = BC$	**2.** ________________
3. $AE = BF$ and $ED = FC$	**3.** Given.
4. ________________	**4.** Substitution postulate.
5. $AD = BC$	**5.** Transitive property (steps 2, 4).

4. *Given:* $\angle SPR \cong \angle QRP$ and $\angle RPQ \cong \angle PRS$
Prove: $\angle SPQ \cong \angle QRS$

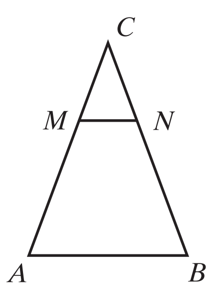


Statements	**Reasons**
1. ______________	**1.** Given.
2. ______________	**2.** If congruent angles are added to congruent angles, the sums are congruent.
3. $\angle SPR + \angle RPQ \cong \angle SPQ$	**3.** ______________
4. ______________	**4.** Partition postulate.
5. $\angle SPQ \cong \angle QRS$	**5.** ______________

In 5–8, in each case write a proof to show that the conclusion is valid.

5. *Given:* $\overline{AC} \cong \overline{BC}$ and $\overline{MC} \cong \overline{NC}$

Prove: $\overline{AM} \cong \overline{BN}$

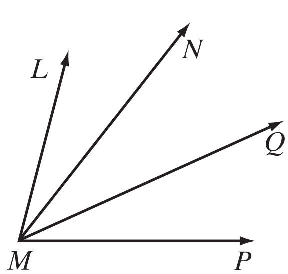


6. *Given:* $\overline{ABCD}$ with $\overline{AB} \cong \overline{CD}$

Prove: $\overline{AC} \cong \overline{BD}$

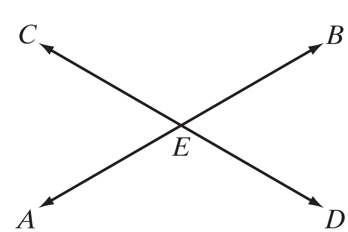
7. *Given:* $\angle LMN \cong \angle PMQ$

Prove: $\angle LMQ \cong \angle NMP$



8. *Given:* $m\angle AEB = 180$ and $m\angle CED = 180$

Prove: $m\angle AEC = m\angle BED$



3-8 THE MULTIPLICATION AND DIVISION POSTULATES

The postulates of multiplication and division are similar to the postulates of addition and subtraction. The postulates in this section are stated in symbols and in words.

The Multiplication Postulate

Postulate 3.8

If $a = b$, and $c = d$, then $ac = bd$.
If equals are multiplied by equals, the products are equal.

When each of two equal quantities is multiplied by 2, we have a special case of this postulate, which is stated as follows:

Postulate 3.9

Doubles of equal quantities are equal.

The Division Postulate

Postulate 3.10

If $a = b$, and $c = d$, then $\frac{a}{c} = \frac{b}{d}$ ($c \neq 0$ and $d \neq 0$).

If equals are divided by nonzero equals, the quotients are equal.

When each of two equal quantities is divided by 2, we have a special case of this postulate, which is stated as follows:

Postulate 3.11

Halves of equal quantities are equal.

Note: Doubles and halves of congruent segments and angles will be handled in Exercise 10.

Powers Postulate

Postulate 3.12

If $a = b$, then $a^2 = b^2$.

The squares of equal quantities are equal.

If $AB = 7$, then $(AB)^2 = (7)^2$, or $(AB)^2 = 49$.

Roots Postulate

Postulate 3.13

If $a = b$ and $a > 0$, then $\sqrt{a} = \sqrt{b}$.

Recall that $\sqrt{a}$ and $\sqrt{b}$ are the positive square roots of a and of b, and so the postulate can be rewritten as:

Postulate 3.13

Positive square roots of positive equal quantities are equal.

If $(AB)^2 = 49$, then $\sqrt{(AB)^2} = \sqrt{49}$, or $AB = 7$.

EXAMPLE 1

Given: $AB = CD$, $RS = 3AB$, $LM = 3CD$

Prove: $RS = LM$

Proof

Statements	***Reasons***
1. $AB = CD$	**1.** Given.
2. $3AB = 3CD$	**2.** Multiplication postulate.
3. $RS = 3AB$	**3.** Given.
4. $RS = 3CD$	**4.** Transitive property of equality (steps 2 and 3).
5. $LM = 3CD$	**5.** Given.
6. $RS = LM$	**6.** Substitution postulate (steps 4 and 5).

EXAMPLE 2

Given: $5x + 3 = 38$

Prove: $x = 7$

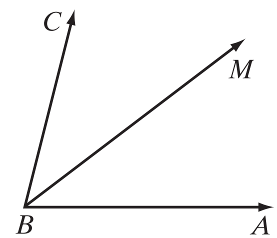
Proof

Statements	***Reasons***
1. $5x + 3 = 38$	**1.** Given.
2. $3 = 3$	**2.** Reflexive property of equality.
3. $5x = 35$	**3.** Subtraction postulate.
4. $x = 7$	**4.** Division postulate.

EXAMPLE 3

Given: $m\angle ABM = \frac{1}{2}m\angle ABC$, $m\angle ABC = 2m\angle MBC$

Prove: $\overrightarrow{BM}$ bisects $\angle ABC$.



Proof

Statements	Reasons
1. $m\angle ABM = \frac{1}{2}m\angle ABC$	**1.** Given.
2. $m\angle ABC = 2m\angle MBC$	**2.** Given.
3. $\frac{1}{2}m\angle ABC = m\angle MBC$	**3.** Division postulate (or halves of equal quantities are equal).
4. $m\angle ABM = m\angle MBC$	**4.** Transitive property of equality.
5. $\angle ABM \cong \angle MBC$	**5.** Congruent angles are angles that have the same measure.
6. $\overrightarrow{BM}$ bisects $\angle ABC$	**6.** The bisector of an angle is a ray whose endpoint is the vertex of the angle and that divides the angle into two congruent angles.

Exercises

Writing About Mathematics

1. Explain why the word "positive" is needed in the postulate "Positive square roots of positive equal quantities are equal."

2. Barry said that "$c \neq 0$" or "$d \neq 0$" but not both could be eliminated from the division postulate. Do you agree with Barry? Explain why or why not.

Developing Skills

In 3 and 4, in each case fill in the missing *statement* or *reason* in the proof to show that the conclusion is valid.

3. *Given:* $AB = \frac{1}{4}BC$ and $BC = CD$

Prove: $CD = 4AB$

Statements	Reasons
1. $AB = \frac{1}{4}BC$	**1.** ________
2. $4 = 4$	**2.** ________
3. $4AB = BC$	**3.** ________
4. $BC = CD$	**4.** ________
5. $4AB = CD$	**5.** ________
6. $CD = 4AB$	**6.** ________

4. *Given:* $m\angle a = 3m\angle b$ and $m\angle b = 20$

Prove: $m\angle a = 60$

Statements	Reasons
1. ________	**1.** Given.
2. ________	**2.** Given.
3. $3m\angle b = 60$	**3.** ________
4. ________	**4.** Transitive property of equality.

In 5–7, in each case write a proof to show that the conclusion is valid.

5. *Given:* $LM = 2MN$ and $MN = \frac{1}{2}NP$

Prove: $\overline{LM} \cong \overline{NP}$

6. *Given:* $2(3a - 4) = 16$

Prove: $a = 4$

7. *Given:* $\overline{PQRS}$, $PQ = 3QR$, and $QR = \frac{1}{3}RS$

Prove: $\overline{PQ} \cong \overline{RS}$

Applying Skills

8. On Monday, Melanie walked twice as far as on Tuesday. On Wednesday, she walked one-third as far as on Monday and two-thirds as far as on Friday. Prove that Melanie walked as far on Friday as she did on Tuesday.

9. The library, the post office, the grocery store, and the bank are located in that order on the same side of Main Street. The distance from the library to the post office is four times the distance from the post office to the grocery store. The distance from the grocery store to the bank is three times the distance from the post office to the grocery store. Prove that the distance from the library to the post office is equal to the distance from the post office to the bank. (Think of Main Street as the line segment $\overline{LPGB}$.)

10. Explain why the following versions of Postulates 3.9 and 3.11 are valid:

Doubles of congruent segments are congruent.	Halves of congruent segments are congruent.
Doubles of congruent angles are congruent.	Halves of congruent angles are congruent.

CHAPTER SUMMARY

Geometric statements can be proved by using **deductive reasoning**. Deductive reasoning applies the laws of logic to a series of true statements to arrive at a conclusion. The true statements used in deductive reasoning may be the given, definitions, postulates, or theorems that have been previously proved.

Inductive reasoning uses a series of particular examples to lead to a general conclusion. Inductive reasoning is a powerful tool in discovering and making **conjectures**. However, inductive reasoning does not prove or explain conjectures; generalizations arising from direct measurements of specific cases are only approximate; and care must be taken to ensure that all relevant examples are examined.

A **proof** using deductive reasoning may be either **direct** or **indirect**. In direct reasoning, a series of statements that include the given statement lead to the desired conclusion. In indirect reasoning, the negation of the desired conclusion leads to a statement that contradicts a given statement.

Postulates

3.1 The Reflexive Property of Equality: $a = a$
3.2 The Symmetric Property of Equality: If $a = b$, then $b = a$.
3.3 The Transitive Property of Equality: If $a = b$ and $b = c$, then $a = c$.
3.4 A quantity may be substituted for its equal in any statement of equality.
3.5 A whole is equal to the sum of all its parts.
3.5.1 A segment is congruent to the sum of all its parts.
3.5.2 An angle is congruent to the sum of all its parts.
3.6 If equal quantities are added to equal quantities, the sums are equal.
3.6.1 If congruent segments are added to congruent segments, the sums are congruent.
3.6.2 If congruent angles are added to congruent angles, the sums are congruent.
3.7 If equal quantities are subtracted from equal quantities, the differences are equal.
3.7.1 If congruent segments are subtracted from congruent segments, the differences are congruent.
3.7.2 If congruent angles are subtracted from congruent angles, the differences are congruent.
3.8 If equals are multiplied by equals, the products are equal.
3.9 Doubles of equal quantities are equal.
3.10 If equals are divided by nonzero equals, the quotients are equal.
3.11 Halves of equal quantities are equal.
3.12 The squares of equal quantities are equal.
3.13 Positive square roots of equal quantities are equal.

VOCABULARY

3-1 Generalization • Inductive reasoning • Counterexample • Conjecture

3-3 Proof • Deductive reasoning • Given • Prove • Two-column proof • Paragraph proof

3-4 Direct proof • Indirect proof • Proof by contradiction

3-5 Postulate • Axiom • Theorem • Reflexive property of equality • Symmetric property of equality • Transitive property of equality • Equivalence relation

3-6 Substitution postulate

3-7 Partition postulate • Addition postulate • Angle addition postulate • Subtraction postulate

3-8 Multiplication postulate • Division postulate • Powers postulate • Roots postulate

REVIEW EXERCISES

In 1–3, in each case: **a.** Write the given definition in a conditional form. **b.** Write the converse of the statement given as an answer to part **a**. **c.** Write the biconditional form of the definition.

1. An obtuse triangle is a triangle that has one obtuse angle.

2. Congruent angles are angles that have the same measure.

3. Perpendicular lines are two lines that intersect to form right angles.

4. Explain the difference between a postulate and a theorem.

5. Name the property illustrated by the following statement:
If $AB = CD$, then $CD = AB$.

6. Is the relation "is greater than" an equivalence relation for the set of real numbers? Explain your answer by demonstrating which (if any) of the properties of an equivalence relation are true and which are false.

In 7–12, in each case draw a figure that illustrates the given information and write a proof to show that the conclusion is valid.

7. *Given:* $\overleftrightarrow{AB}$ bisects $\overline{CD}$ at M.
Prove: $CM = MD$

8. *Given:* $\overline{RMST}$ is a line segment, $\overline{RM} \cong \overline{MS}$, and $\overline{MS} \cong \overline{ST}$.
Prove: $\overline{RM} \cong \overline{ST}$

9. *Given:* $\overline{ABCD}$ is a line segment and $\overline{AC} \cong \overline{BD}$.
Prove $\overline{AB} \cong \overline{CD}$

10. *Given:* $\overline{SQRP}$ is a line segment and $SQ = RP$.
Prove: $SR = QP$

11. *Given:* $\overrightarrow{BC}$ bisects $\angle ABD$ and $m\angle CBD = m\angle PQR$.

Prove: $m\angle ABC = m\angle PQR$

12. *Given:* $\overline{CD}$ and $\overline{AB}$ bisect each other at E and $CE = BE$.

Prove: $CD = AB$

13. A student wrote the following proof:

Given: $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \perp \overline{BC}$

Prove: $\overline{CD} \perp \overline{BC}$

Statements	***Reasons***
1. $\overline{AB} \cong \overline{CD}$	**1.** Given.
2. $\overline{AB} \perp \overline{BC}$	**2.** Given.
3. $\overline{CD} \perp \overline{BC}$	**3.** Substitution postulate.

What is the error in this proof?

14. A **palindrome** is a sequence of numbers or letters that reads the same from left to right as from right to left.

a. Write the definition of a palindrome as a conditional statement.

b. Write the converse of the conditional statement in **a**.

c. Write the definition as a biconditional.

Exploration

The following "proof" leads to the statement that twice a number is equal to the number. This would mean, for example, that if $b = 1$, then $2(1) = 1$, which is obviously incorrect. What is the error in the proof?

Given: $a = b$

Prove: $b = 2b$

Statements	***Reasons***
1. $a = b$	**1.** Given.
2. $a^2 = ab$	**2.** Multiplication postulate.
3. $a^2 - b^2 = ab - b^2$	**3.** Subtraction postulate.
4. $(a + b)(a - b) = b(a - b)$	**4.** Substitution postulate.
5. $a + b = b$	**5.** Division postulate.
6. $b + b = b$	**6.** Substitution postulate.
7. $2b = b$	**7.** Substitution postulate.

CUMULATIVE REVIEW CHAPTERS 1–3

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If $y = 2x - 7$ and $y = 3$, then x is equal to
(1) 1 (2) -1 (3) 5 (4) -5

2. The property illustrated in the equality $2(a + 4) = 2(4 + a)$ is
(1) the distributive property.
(2) the associative property.
(3) the identity property.
(4) the commutative property.

3. In biconditional form, the definition of the midpoint of a line segment can be written as
(1) A point on a line segment is the midpoint of that segment if it divides the segment into two congruent segments.
(2) A point on a line segment is the midpoint of that segment if it divides the segment into two congruent segments.
(3) A point on a line segment is the midpoint of that segment only if it divides the segment into two congruent segments.
(4) A point on a line segment is the midpoint of that segment if and only if it divides the segment into two congruent segments.

4. The multiplicative identity element is
(1) 1 (2) -1 (3) 0 (4) not a real number

5. The angle formed by two opposite rays is
(1) an acute angle.
(2) a right angle.
(3) an obtuse angle.
(4) a straight angle.

6. The inverse of the statement "If two angles are right angles, then they are congruent" is
(1) If two angles are not congruent, then they are not right angles.
(2) If two angles are not right angles, then they are not congruent.
(3) If two angles are congruent, then they are right angles.
(4) Two angles are not right angles if they are not congruent.

7. The statements "Today is Saturday or I go to school" and "Today is not Saturday" are both true statements. Which of the following statements is also true?
(1) Today is Saturday.
(2) I do not go to school.
(3) I go to school.
(4) Today is Saturday and I go to school.

8. $\overline{ABCD}$ is a line segment and B is the midpoint of $\overline{AC}$. Which of the following must be true?

(1) C is the midpoint of $\overline{BD}$
(2) $AB = BC$
(3) $AC = CD$
(4) $AC + BD = AD$

9. The statements "$AB = BC$" and "$DC = BC$" are true statements. Which of the following must also be true?

(1) $AB + BC = AC$
(2) A, B, and C are collinear
(3) B, C, and D are collinear
(4) $AB = DC$

10. Triangle LMN has exactly two congruent sides. Triangle LMN is

(1) a right triangle.
(2) a scalene triangle.
(3) an isosceles triangle.
(4) an equilateral triangle.

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Give a reason for each step used in the solution of the equation.

$3x + 7 = 13$	*Given*
$-7 = -7$	________
$3x \quad = 6$	________
$x \quad = 2$	________

12. *Given:* $\overline{DE} \perp \overline{EF}$

Prove: $\triangle DEF$ is a right triangle.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. *Given:* $\triangle ABC$ with D a point on $\overline{AB}$ and $AC = AD + DB$.

Prove: $\triangle ABC$ is isosceles.

14. $\overline{PQR}$ is a line segment. $PQ = 4a - 3$, $QR = 3a + 2$ and $PR = 8a - 6$. Is Q the midpoint of $\overline{PQR}$? Justify your answer.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. $\overrightarrow{BD}$ bisects $\angle ABC$, $m\angle ABD = 3x + 18$ and $m\angle DBC = 5x - 30$. If $m\angle ABC = 7x + 12$, is $\angle ABC$ a straight angle? Justify your answer.

16. For each statement, the hypothesis is true. Write the postulate that justifies the conclusion.

a. If $x = 5$, then $x + 7 = 5 + 7$.

b. If $2y + 3$ represents a real number, then $2y + 3 = 2y + 3$.

c. If $\overline{RST}$ is a line segment, then $\overline{RS} + \overline{ST} = \overline{RT}$.

d. If $y = 2x + 1$ and $y = 15$, then $2x + 1 = 15$.

e. If $a = 3$, then $\frac{a}{5} = \frac{3}{5}$.